JOURNAL OF MATERIALS & CONSTRUCTION

Journal of Materials and Construction
ISSN 2734-9438
Website: www. jomc.vn

# Quantification of accuracy degradation in differential leveling under unfavorable terrain conditions

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KEYWORDS

Differential leveling
Accuracy degradation
Geometric efficiency
Offset configuration
Atmospheric refraction

ABSTRACT

Differential leveling is a fundamental surveying technique valued for its simplicity and high intrinsic precision. The method's accuracy relies on the midpoint principle, wherein the level instrument is placed equidistant between backsight and foresight staffs to eliminate distance-dependent systematic errors. However, in mountainous or complex terrains, this ideal configuration is often impractical. The instrument must then be set at a perpendicular offset from the baseline, which increases sight distances and error propagation. This study investigates the mechanisms of accuracy degradation under such unfavorable conditions, focusing on the geometric elongation of the optical sight path. We analytically examine the principal distance-dependent error sources, including collimation error, vertical index error, Earth curvature, atmospheric refraction, and staff-reading uncertainty, and show how they scale with sight length. Using the geometric relationship between the ideal baseline and actual sight lengths, we derive an accuracy degradation coefficient K to quantify the loss of efficiency caused by offset instrument placement. The coefficient is evaluated over a range of offset ratios ($0.05 \le h/S \le 2.00$), revealing a pronounced non-linear decrease in accuracy with increasing offset. The results indicate that degradation remains minor for $h/S < 0.25$, becomes significant for moderate offsets ($0.25 < h/S < 0.9$), and is severe when $h/S > 1.5$. Based on these findings, we propose practical guidelines: applying double-station leveling in moderate-offset conditions and resorting to alternative height-determination methods in extreme terrain. The proposed degradation coefficient provides a simple tool for survey planning and quality control, enabling engineers to assess expected accuracy loss and design appropriate observation strategies for differential leveling in challenging topographic environments.

## 1. Introduction

Differential leveling determines elevation differences between points on the Earth's surface by means of a horizontal line of sight [1]. Owing to its simplicity and high intrinsic precision, this method remains the primary technique for establishing vertical control networks and is widely applied in subsidence monitoring for industrial areas, hydraulic engineering works, and transportation infrastructure [2].

The fundamental principle of high-precision leveling requires that the instrument be positioned at equal distances from the backsight (BS) and foresight (FS) staffs [1]. Ideally, to minimize the total sight length and reduce atmospheric effects, the instrument is placed directly on the straight line connecting the two points (Figure 1) [1]. This configuration, commonly referred to as the midpoint method, effectively cancels both instrumental errors (such as collimation error) and natural systematic errors, including Earth curvature and atmospheric refraction [1].

However, in mountainous terrain or complex construction environments, this ideal collinear setup is often impractical. In such cases, surveyors are forced to place the instrument at a perpendicular offset from the baseline connecting the two staffs, resulting in an isosceles triangle geometry. This deviation increases the sight distances to both staffs, thereby amplifying random errors as well as systematic errors that are proportional to distance [3].

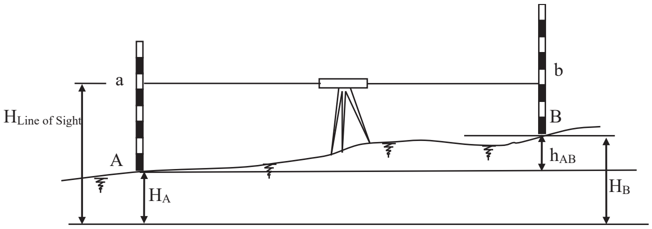

**Figure 1.** Midpoint method in differential leveling

Although the adverse influence of offset instrument placement on leveling accuracy is qualitatively well recognized, its quantitative impact has not been sufficiently addressed in practical guidelines. Therefore, this study aims to analyze the principal error sources associated with offset configurations and to derive a mathematical Degradation Coefficient ($K$) that characterizes the loss of accuracy as a function of instrument geometry. The proposed coefficient provides a practical tool for technical staff to evaluate observation schemes and to

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Received 28/01/2026, revised 23/03/2026, accepted 26/03/2026
Link DOI: https://doi.org/10.54772/jomc.v16i01.1235

ensure the required accuracy in challenging field conditions.

Unlike existing qualitative guidelines, this study provides an analytical and quantitative framework to evaluate accuracy degradation in differential leveling under offset configurations. The proposed degradation coefficient offers a simple yet rigorous metric that can be directly applied during survey planning, thereby bridging the gap between classical leveling theory and practical field decision-making.

## 2. Materials and methods

### *2.1. Analysis of error sources in differential leveling*

To quantify the degradation of leveling accuracy under offset instrument setups, the primary error sources that depend on sight distance are first identified and analyzed.

#### *2.1.1. Collimation error (line-of-sight error)*

Ideally, the optical axis of a level instrument should be perfectly parallel to the axis of the bubble tube or the reference plane of the compensator. In practice, however, a small residual vertical angular error, denoted by $\varepsilon$, always exists (Figure 2) [3]. This angular deviation produces a height error that increases linearly with the sight distance $d$. The resulting elevation error $\Delta h_1$ can be expressed as:

$$\Delta h_1 = \mathrm{d}\ \varepsilon \ (1)$$

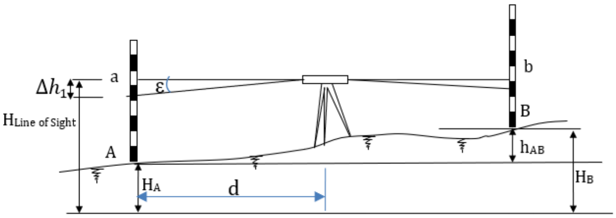

**Figure 2.** Error caused by a non-horizontal line of sight in the midpoint differential leveling method.

Although modern automatic levels significantly reduce this effect through compensator mechanisms, a residual error remains. Under ideal midpoint configurations, this error is largely cancelled by equal backsight (BS) and foresight (FS) distances. However, when terrain constraints force an increase in $d$, even small imbalances between BS and FS distances can cause a noticeable amplification of this error [4].

#### *2.1.2. Instrument collimation and vertical index errors*

For a leveling instrument, the line of sight should ideally be parallel to the axis of the tubular bubble or the reference plane of the compensator. In practice, a small angular misalignment always exists between these axes, commonly referred to as the vertical index error, denoted by $i$. This error causes the line of sight to deviate from the true horizontal direction, producing an effect equivalent to a collimation error [3].

The resulting height error is proportional to the sight distance $d$ and can be expressed as:

$$\Delta h_2 = \mathrm{d}\ \mathrm{i} \ (2)$$

Similar to collimation error, the influence of the index error can be effectively eliminated by applying the midpoint leveling technique, in which the backsight and foresight distances are equal. However, in practical field conditions, variations in instrument adjustment and unavoidable inequalities between sight distances cause the effect of $i$ to persist. Consequently, the vertical index error remains a non-negligible source of systematic error in high-precision leveling, particularly when long or unbalanced sight distances are involved.

#### *2.1.3. Earth curvature and atmospheric refraction*

In differential leveling, the level surface follows a curve approximately parallel to the geoid, whereas the line of sight is a tangent to the equipotential surface and is further bent downward due to atmospheric refraction. The combined effect of Earth curvature and refraction produces a systematic height error $\Delta h_3$ that is proportional to the square of the sight distance:

$$\Delta h_3 = \frac{(1-k)\, d^2}{2R} \ (3)$$

where

$R$ is the mean radius of the Earth, and

$k$ is the atmospheric refraction coefficient

When the backsight and foresight distances are equal ($d_{BS} = d_{FS}$), this error is effectively cancelled (Figure 3). However, in offset configurations, the increased sight lengths and differing atmospheric paths may result in incomplete cancellation, particularly due to spatial and temporal variations in atmospheric refraction [3,4].

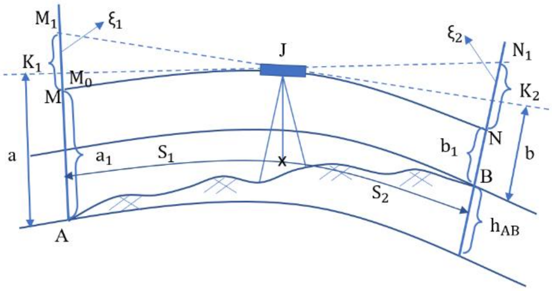

**Figure 3.** Effect of Earth curvature and vertical atmospheric refraction on leveling measurements [5].

#### *2.1.4. Reading and pointing errors*

The precision of reading a leveling staff depends on the telescope magnification $V$ and the sight distance $d$. The standard deviation of the reading error is approximately proportional to the sight distance and inversely proportional to the magnification. This relationship can be expressed as:

$$\Delta h_4 = \frac{C\, d}{V} \ (4)$$

where $C$ is an empirical constant determined by observational conditions and instrument characteristics [3].

This error is random in nature but increases directly as the sight path lengthens, which commonly occurs in unfavorable terrain where offset setups are unavoidable.

In summary, even under ideal conditions all the above systematic errors exist. Differential levelling practice (short sights, balanced BS/FS, reciprocal runs) aims to cancel many systematic terms [4]. However, in rough terrain one cannot always place the instrument on the midline, which introduces additional uncompensated error as discussed next.

### *2.2. Geometric model of accuracy degradation*

In complex or mountainous terrain, the leveling instrument cannot always be positioned on the straight line connecting the two measurement points. Let:

- $S$ denote the horizontal baseline distance between the two leveling points
- $h$ denote the perpendicular offset distance of the instrument from the baseline $S$

Under this configuration, the actual sight lengths $d'_1$ and $d'_2$ correspond to the hypotenuse of the right-angled triangles formed by the offset geometry (Figure 4).

In the ideal midpoint setup, the total optical path length equals the baseline $S$. In contrast, for the offset configuration, the total optical path length becomes:

$$d'_1 = d'_2 = \sqrt{\left(\frac{S}{2}\right)^2 + h^2} \quad (5)$$

The geometric increase in path length, $\Delta S$, can therefore be written as:

$$\Delta S = (d'_1 + d'_2) - S \quad (6)$$

Since most leveling errors are functions of the sight distance $d$, an increase in total path length directly leads to an increase in the variance of the observed height difference. To characterize this effect, the degradation of accuracy is represented by the ratio between the ideal and actual optical paths.

Accordingly, the accuracy degradation coefficient ($K$) is defined as:

$$K = \frac{S}{d'_1 + d'_2} \quad (7)$$

Or in terms of the offset ratio (h/S):

$$K = \frac{1}{\sqrt{1+4\left(\frac{h}{S}\right)^2}} \quad (8)$$

By definition, $K \leq 1$. This coefficient represents the geometric efficiency of the leveling setup. A smaller value of $K$ indicates a longer optical path relative to the baseline, implying reduced measurement precision and increased susceptibility to distance-dependent errors and atmospheric instability.

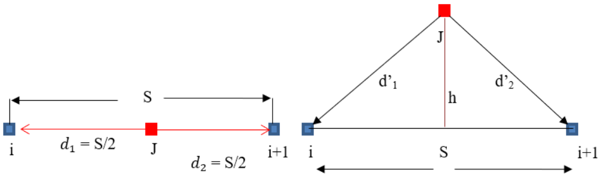

**Figure 4.** Schematic of level instrument setup under difficult terrain conditions.

The proposed geometric model assumes symmetric sight lengths and neglects second-order atmospheric turbulence effects, which is consistent with standard assumptions in high-precision leveling analysis.

## 3. Results and discussion

The accuracy degradation coefficient ($K$) was simulated and computed for a range of terrain configurations characterized by the offset ratio ($h/S$). The ratio varied from 0.05 (5 %) to 2.00 (200 %), and the results are presented in Table 1 and visualized in Figure 5.

**Table 1.** Accuracy degradation coefficient ($K$) as a function of offset ratio ($h/S$).

| No | h/S | K | No | h/S | K | No | h/S | K | No | h/S | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.05 | 1.00 | 11 | 0.55 | 0.67 | 21 | 1.05 | 0.43 | 31 | 1.55 | 0.31 |
| 2 | 0.10 | 0.98 | 12 | 0.60 | 0.64 | 22 | 1.10 | 0.41 | 32 | 1.60 | 0.30 |
| 3 | 0.15 | 0.96 | 13 | 0.65 | 0.61 | 23 | 1.15 | 0.40 | 33 | 1.65 | 0.29 |
| 4 | 0.20 | 0.93 | 14 | 0.70 | 0.58 | 24 | 1.20 | 0.38 | 34 | 1.70 | 0.28 |
| 5 | 0.25 | 0.89 | 15 | 0.75 | 0.55 | 25 | 1.25 | 0.37 | 35 | 1.75 | 0.27 |
| 6 | 0.30 | 0.86 | 16 | 0.80 | 0.53 | 26 | 1.30 | 0.36 | 36 | 1.80 | 0.27 |
| 7 | 0.35 | 0.82 | 17 | 0.85 | 0.51 | 27 | 1.35 | 0.35 | 37 | 1.85 | 0.26 |
| 8 | 0.40 | 0.78 | 18 | 0.90 | 0.49 | 28 | 1.40 | 0.34 | 38 | 1.90 | 0.25 |
| 9 | 0.45 | 0.74 | 19 | 0.95 | 0.47 | 29 | 1.45 | 0.33 | 39 | 1.95 | 0.25 |
| 10 | 0.50 | 0.71 | 20 | 1.00 | 0.45 | 30 | 1.50 | 0.32 | 40 | 2.00 | 0.24 |

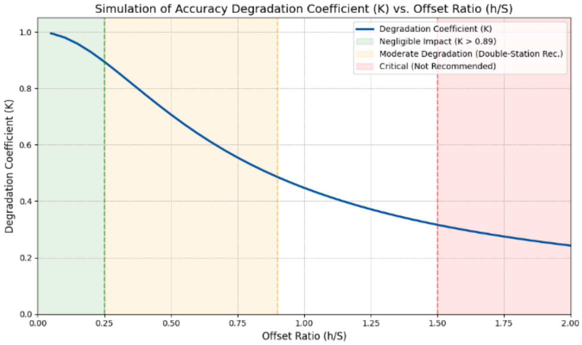

**Figure 5.** Simulated relationship between the degradation coefficient $K$ and the offset ratio $(h/S)$.

The simulation results exhibit a clear monotonic and non-linear decrease of the coefficient $K$ with increasing h/S. For small offsets, $K$ remains close to unity, indicating that the geometric efficiency of the leveling configuration is nearly equivalent to the ideal midpoint setup. As the offset increases, the total optical path length grows rapidly, leading to a pronounced reduction in $K$ and, consequently, in effective measurement precision.

Based on the observed trends in Figure 5, three characteristic offset ranges can be identified for interpretation purposes:

### *3.1. Low-offset range* $(h/S < 0.25)$

Corresponding to the green-shaded region, the coefficient remains above $K \approx 0.89$. In this range, the slope of the curve is relatively gentle, indicating that the geometric efficiency of the leveling configuration is largely preserved. The associated increase in distance-dependent errors is minor, and standard midpoint or near-midpoint setups can be expected to meet typical engineering accuracy requirements without additional corrective measures.

### *3.2. Moderate-offset range* $(0.25 < \frac{h}{S} < 1.5)$

As shown by the yellow-shaded region, the degradation coefficient decreases rapidly with increasing offset. This portion of the curve exhibits a pronounced change in slope, indicating heightened sensitivity to geometric elongation of the sight path. Small increases in $h/S$ within this range lead to disproportionate reductions in $K$, implying a rapid amplification of both systematic and random errors. This behavior explains why single-station leveling becomes increasingly unreliable under moderate terrain constraints and supports the recommendation to apply double-station or reciprocal leveling techniques to mitigate residual errors.

### *3.3. High-offset range* $(h/S > 1.5)$

This range is represented by the red-shaded region, where the coefficient falls below $K \approx 0.30$. In this regime, the optical path length is substantially greater than the baseline, and the leveling configuration becomes highly inefficient. The shallow gradient of the curve in this range indicates that further increases in offset yield diminishing geometric benefit while exposing the measurements to severe atmospheric instability, particularly due to refraction effects along long, low-grazing sight paths. Under such conditions, geometric leveling is unlikely to deliver reliable results, and alternative height determination methods should be considered.

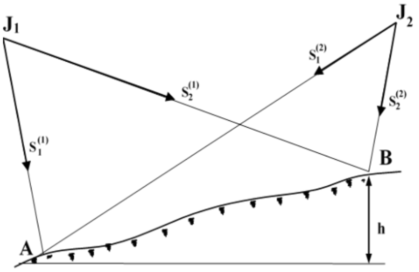

**Figure 6.** Schematic of the double-run (double-station) geometric leveling method.

Overall, the simulation visually confirms the analytical findings presented earlier and provides an intuitive interpretation of the proposed degradation coefficient. The clear separation of offset regimes in Figure 6 highlights the practical value of $K$ as a planning and decision-support indicator, enabling surveyors to anticipate accuracy loss and to select appropriate observation strategies under challenging terrain conditions.

### *3.4. Mitigation strategies*

To mitigate accuracy loss in the moderate-offset range, a double-station leveling strategy is suggested (Figure 6). In this approach, observations are conducted from two approximately symmetrical instrument positions ($J_1$ and $J_2$) relative to the baseline. Averaging the results from both setups helps reduce residual systematic effects, including compensator drift and refraction asymmetry. For extreme offset conditions ($h/S > 1.5$), geometric leveling becomes increasingly inefficient. In such cases, alternative height determination methods, such as trigonometric leveling using a high-precision Total Station, should be considered.

The computed values of $K$ for $0.05 \leq h/S \leq 2.00$ are given in Table 1. These results are consistent with the simple theoretical model and illustrate the non-linear dependence of accuracy on geometry. In the ideal case ($h = 0$), $K = 1$ and no degradation occurs. For moderate offsets ($h/S \approx 0.3$–$0.5$), the coefficient drops to $\sim 0.7$–$0.8$, indicating that errors grow by 20–30 %. For large offsets ($h/S > 1.0$), the degradation is severe. Thus, the data justify the rule of thumb that the instrument height above the line should be kept below 25 % of the separation distance; beyond that, special measures are needed.

Several corrective strategies are suggested by these findings.

First, one should minimize $h$ by careful instrument placement (e.g. using a longer tripod or alternate survey route) whenever terrain allows. Second, if the instrument cannot be centered, one should perform reciprocal or double-run levelling. In this approach the same two points are levelled from two opposite setups (J1 and J2 in Figure 6) so that each error source has a chance to cancel out when averaging the results. This technique is standard in high-precision geodetic levelling [4]. Third, explicit correction for curvature and refraction can be applied. Modern practice uses atmospheric profiles to adjust rod readings [7]. Finally, rigorous quality control is essential: repeated loops and misclosure checks are used to detect any unmodeled bias [6]. For example, the California field tests reported by Castle et al. showed that uncorrected refraction errors tended to be masked by other survey errors, and only after discarding poor sections did a systematic bias become evident [7].

Overall, the theoretical model aligns well with field experience. Other authors have noted that “atmospheric refraction is one of the largest errors in leveling” [7] and our analysis shows that it increases with sight length. Similarly, quality control manuals emphasize that first-order levelling must follow strict specifications (balanced sights, calibrated rods, loop closures) to achieve millimeter-level precision [4,8-9]. The proposed coefficient $K$ is a practical metric: by computing $K$ for planned setups, surveyors can predict the loss of precision and decide whether to shorten sights or adjust the plan. A simple simulation of levelling errors using the above formulas confirms that the height uncertainty grows roughly in inverse proportion to $K$.

## 4. Validation and sensitivity analysis

### *4.1. Internal consistency and theoretical validation*

The validity of the proposed degradation coefficient $K$ was first examined through internal consistency checks and theoretical expectations. By formulation (7).

In the ideal midpoint configuration ($h = 0$), the actual sight distances reduce to $d_1' = d_2' = S/2$, yielding $K = 1.0$. This result confirms that the coefficient correctly reproduces the optimal geometric condition with no accuracy loss. As the offset $h$ increases, both $d_1'$ and $d_2'$ increase monotonically, leading to a strictly decreasing $K$. This behavior is consistent with the known distance-dependent nature of leveling errors and confirms the physical plausibility of the proposed metric.

Furthermore, the non-linear decrease of $K$ with respect to the ratio $h/S$ reflects the quadratic growth of sight distances inherent to the triangular geometry. This validates that the coefficient captures the essential geometric mechanism responsible for accuracy degradation.

### *4.2. Sensitivity of the degradation coefficient to instrument offset*

To assess the sensitivity of the proposed coefficient, the rate of change of $K$ with respect to the offset ratio $h/S$ was analyzed. The results presented in Table 1 indicate that $K$ is relatively insensitive to small offsets ($h/S < 0.2$), where incremental increases in $h$ produce only marginal reductions in accuracy.

However, as the offset ratio exceeds approximately 0.3, the slope of the $K(h/S)$ curve increases significantly, indicating heightened sensitivity. In this regime, small additional offsets result in disproportionately large losses in effective precision. This sensitivity behavior explains why minor deviations from the midpoint configuration are often tolerable in practice, whereas larger offsets rapidly compromise measurement reliability.

The sensitivity analysis thus supports the classification of offset regimes discussed in Section 3 and provides a quantitative basis for identifying threshold values relevant to field operations.

### *4.3. Robustness with respect to error source aggregation*

Although the degradation coefficient $K$ is derived purely from geometric considerations, its practical relevance depends on how well it reflects the combined influence of dominant leveling error sources. Most systematic and random errors in geometric leveling, including collimation error, vertical index error, curvature and refraction effects, and staff-reading uncertainty, are proportional to the sight distance $d$ or its square.

Because $K$ represents the ratio between the ideal and actual total optical path lengths, it acts as a first-order scaling factor for the aggregated variance of distance-dependent errors. Consequently, the coefficient remains robust with respect to different combinations of error sources, instrument types, and observational conditions, provided that distance remains the governing parameter.

This property allows the proposed framework to be applied without requiring explicit modeling of each individual error component, thereby enhancing its practical applicability.

### *4.4. Comparison with conventional field practices*

Conventional leveling guidelines emphasize maintaining short and balanced sight distances but rarely provide a quantitative measure of acceptable deviation from the midpoint configuration. The proposed degradation coefficient complements these guidelines by offering a simple numerical indicator that can be computed during survey planning.

For example, the commonly adopted rule of limiting instrument offset to approximately one-quarter of the baseline length corresponds to $K \approx 0.89$, which aligns well with the empirical experience that accuracy degradation becomes noticeable beyond this point. Similarly, the frequent recommendation to apply double-run or reciprocal leveling in difficult terrain is consistent with the observed rapid decrease of $K$ in the moderate-offset range.

This agreement between the proposed metric and established field practice provides indirect validation of the coefficient and supports its practical relevance.

### *4.5. Limitations and applicability*

It should be noted that the proposed degradation coefficient does not explicitly account for temporal atmospheric variability, local turbulence, or instrument-specific compensator dynamics. Instead, these effects are implicitly captured through their dependence on sight distance.

As a result, the coefficient is most effective as a planning and diagnostic tool rather than as a replacement for rigorous error modeling. In extreme atmospheric conditions or for ultra-high-precision applications, additional correction strategies may still be required.

As the proposed coefficient is derived from deterministic geometric relationships, its validation relies primarily on theoretical consistency and sensitivity analysis rather than on site-specific field measurements.

### *4.6. Summary of validation results*

Overall, the validation and sensitivity analyses demonstrate that the degradation coefficient $K$ is:

- Theoretically consistent, reproducing known limiting cases;
- Sensitive to critical geometric changes, particularly in moderate to large offset configurations;
- Robust with respect to aggregated distance-dependent error sources;
- Consistent with established surveying practice, despite offering a new quantitative interpretation.

These properties confirm that the proposed coefficient provides a reliable and practical framework for assessing and managing accuracy degradation in geometric leveling under challenging terrain conditions.

## 5. Conclusion

Unfavorable terrain conditions can significantly degrade the accuracy of geometric leveling by increasing sight distances and amplifying distance-dependent errors. This study quantified the effect through the introduction of a degradation coefficient $K$, which reflects the geometric efficiency of a given instrument setup.

From a practical perspective, the following conclusions can be drawn:

- Optimization: Instrument placement should prioritize minimizing the perpendicular offset $h$, with the geometric midpoint remaining the optimal configuration.
- Pre-assessment: In difficult terrain, the ratio $h/S$ should be estimated in advance to assess the expected degradation using Table 1 or the simulation graph.
- Operational guidance:
  - For $h/S < 0.25$, accuracy degradation is minor and special corrective measures are typically unnecessary.
  - For $0.25 < h/S < 0.9$, double-run or double-station leveling strategies are recommended to maintain accuracy.
  - For $h/S > 1.5$, geometric leveling is likely unsuitable; alternative measurement techniques (e.g., trigonometric leveling) should be considered.

These precautions, supported by existing standards and experiments [4,6], will ensure that differential levelling meets the required accuracy even in challenging environments.

Beyond accuracy assessment, the proposed framework can be readily implemented as a planning and decision-support tool for designing leveling schemes in complex terrain.

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