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Finite difference method with laplace transform technique for vertical drains

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KEYWORDS

PVD
Finite difference method
Laplace transform technique

ABSTRACT

The paper present finite diference method (FDM) with Laplace transform technique for vertical drains. This method is capable of simulation of multi loading with time and complex boundary conditions. The Laplace transform technique is also ultilized in order to simplify the derivative function of time to an algebra function. The discreted function then can easily determine the pore pressure at each points. The pore pressure and average consolidation degree by the present solution shows good agreement with previous solutions in three differences load applying: instant loading, single ramp loading and multi-ramp loading. This paper also proposes a discreted 10 points is a practical application for multi-ramp loading with the error in predicting average consolidation degree less than 5 %. In order to verify the efficient of the present solution, a real embankment treated by vertical drains with vacuum consolidation technique are analyzed. The pore pressures and settlements have been compared with field data and previous researchs.

1-Introduction

Vertical drains have been analyzed by differences methods including semi-analytical method, Laplace transform merhod, Finite elements method (FEM). There have been extensive studies [3],[6-10],[14],[11-12],[17-18]. However there was only Onoue's research (1988) considered about finite difference method in vertical drains. But the finite difference method (FDM) required two dimension point grids therefore it become much complicate for application. This paper present a FDM with Laplace transform technique in one dimension. This method also consider as the method of line because it will take account of time variant in each discreted points. The analysis results will be compared with previous reseach values. This method will be verified with the field data from Rujikiatkamjorn et al (2008)[16] analysis results of Tianjin Port, China.

2. Basic equation

The basic model for FDM shows in Figure. 1. Some denoted parameters are H = thickness of soil; r_w = radius of the drain; r_s = radius of smeared zone; r_e = equivalent radius of the influence zone; k_h = horizontal coefficient of permeability of natural soil; k_v = vertical coefficient of permeability of the soil; m_v = coefficient of volume compressibility of the soil; k_s = horizontal coefficient of permeability of the smeared zone; k_w = coefficient of permeability of the vertical drain. $\sigma(t)$ = the surcharge load with time, N = is the number of discreted points, $\Delta z = H/(N+1)$ is the discreted spacing.

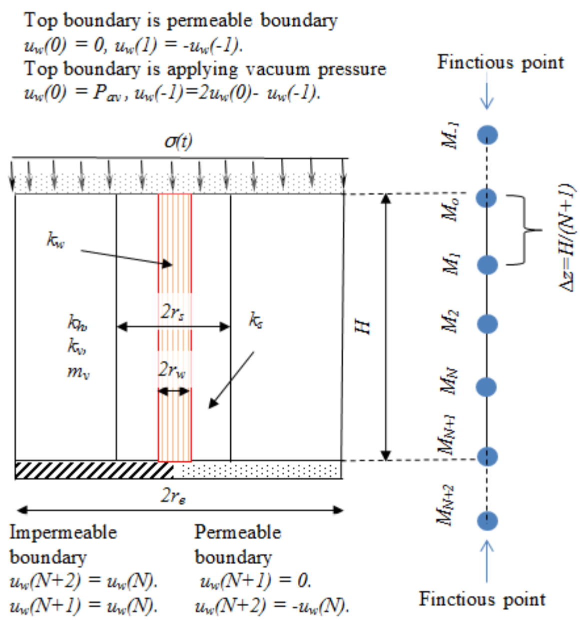

Figure 1. Skematic model for finite difference method with Laplace transform technique for vertical drains.

M_i = is discreted point at location i; discretizing element into N points (the first point M_0, the final point M_{N+1}); in order to obtain solution of differential 4 order equation, it is needed to divide two more finctious points are M_{-1} and M_{N+2}. Pore pressure at location i is $u_w(i)$.

The general partial differential consolidation equation have been derived in Tang và Onitsuka (2000)[14]

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$$\frac{c_v}{\varphi_2}\frac{\partial^4 u_w(z,t)}{\partial z^4}-\frac{1}{\varphi_2}\frac{\partial^3 u_w(z,t)}{\partial z^2 \partial t}-\frac{c_h\varphi_1}{\varphi_2}\frac{\partial^2 u_w(z,t)}{\partial z^2}+\frac{\partial u_w(z,t)}{\partial t}=\frac{\partial \sigma(t)}{\partial t} \quad (1)$$

The relation of pore pressure in the drain and average pore pressure of soil is also derived.

$$\frac{\partial^2 u_w(z,t)}{\partial z^2}=-\varphi_2(\bar{u}_1(z,t)-u_w(z,t))=0 \quad (2)$$

Where

$$c_v=\frac{k_v}{m_v\gamma_w}, c_h=\frac{k_h}{m_v\gamma_w}, s=\frac{r_s}{r_w}, n=\frac{r_e}{r_w}$$

$$\varphi_1=\frac{2}{r_e^2 F}\left[1+(n^2-1)\frac{k_v}{k_w}\right], \varphi_2=(n^2-1)\frac{2}{F}\frac{k_h}{k_w}\left(\frac{1}{r_e}\right)^2$$

$$F=\left(\ln\frac{n}{s}+\frac{k_h}{k_s}\ln s-\frac{3}{4}\right)\frac{n^2}{n^2-1}+\frac{s^2}{n^2-1}\left(1-\frac{s^2}{4n^2}\right)+\frac{k_h}{k_s}\frac{1}{n^2-1}\left(1-\frac{1}{4n^2}\right)$$

$u_w(z,t)$ = pore pressure of the drain

$\bar{u}_1(z,t)$ = average pore pressure of soil.

$\sigma(t)$ = the surcharge load.

Appling Laplace transform technique to (1) yeild.

$$\frac{c_v}{\varphi_2}\frac{\partial^4 Lu_w(z,s)}{\partial z^4}-\frac{(c_h\varphi_1+s)}{\varphi_2}\frac{\partial^2 Lu_w(z,s)}{\partial z^2}+sLu_w(z,s)-u_o=L\left(\frac{\partial\sigma(t)}{\partial t}\right) \quad (3)$$

where

$Lu_w(z,s)$ = the Laplace transform of $u_w(z,t)$

u_o = the initial pore pressure disstribution in soil

$L\left(\frac{\partial\sigma(t)}{\partial t}\right)$ = Laplace transform of $\frac{\partial\sigma(t)}{\partial t}$

Carry out the same transform technique to equation (2) to obtain

$$\frac{\partial^2 Lu_w(z,s)}{\partial z^2}+\varphi_2 L\bar{u}_1(z,s)-\varphi_2 Lu_w(z,s)=0 \quad (4)$$

Where $L\bar{u}_1(z,s)$ is Laplace transform of $\bar{u}_1(z,s)$

Applying Taylor series at location z_i with $U_i = Lu_w[i]$ (Laplace transform pore pressure value at point i) and omitting the small value of fifth order Δz^5:

$$Lu_w[z_i+\Delta z,s]=U_{i+1}\approx U_i+\Delta z\frac{\partial U}{\partial z}+\frac{\Delta z^2}{2!}\frac{\partial^2 U}{\partial z^2}+\frac{\Delta z^3}{3!}\frac{\partial^3 U}{\partial z^3}+\frac{\Delta z^4}{4!}\frac{\partial^4 U}{\partial z^4} \quad (5)$$

$$Lu_w[z_i-\Delta z,s]=U_{i-1}\approx U_i-\Delta z\frac{\partial U}{\partial z}+\frac{\Delta z^2}{2!}\frac{\partial^2 U}{\partial z^2}-\frac{\Delta z^3}{3!}\frac{\partial^3 U}{\partial z^3}+\frac{\Delta z^4}{4!}\frac{\partial^4 U}{\partial z^4} \quad (6)$$

$$Lu_w[z_i+2\Delta z,s]=U_{i+2}\approx U_i+2\Delta z\frac{\partial U}{\partial z}+\frac{4\Delta z^2}{2!}\frac{\partial^2 U}{\partial z^2}+\frac{8\Delta z^3}{3!}\frac{\partial^3 U}{\partial z^3}+\frac{16\Delta z^4}{4!}\frac{\partial^4 U}{\partial z^4} \quad (7)$$

$$Lu_w[z_i-2\Delta z,s]=U_{i+2}\approx U_i-2\Delta z\frac{\partial U}{\partial z}+\frac{4\Delta z^2}{2!}\frac{\partial^2 U}{\partial z^2}-\frac{8\Delta z^3}{3!}\frac{\partial^3 U}{\partial z^3}+\frac{16\Delta z^4}{4!}\frac{\partial^4 U}{\partial z^4} \quad (8)$$

The sum of (5) and (6) yeild

$$U_{i+1}+U_{i-1}=2U_i+\Delta z^2\frac{\partial^2 U}{\partial z^2}+\frac{\Delta z^4}{12}\frac{\partial^4 U}{\partial z^4} \quad (9)$$

The sum (7) and (8) yeild

$$U_{i+2}+U_{i-2}=2U_i+4\Delta z^2\frac{\partial^2 U}{\partial z^2}+\frac{4\Delta z^4}{3}\frac{\partial^4 U}{\partial z^4} \quad (10)$$

Solve (9) and (10) to obtain the differential functions:

$$\frac{\partial^2 U}{\partial z^2}=\frac{-1}{12}\left(\frac{30U_i+U_{i-2}-16U_{i-1}-16U_{i+1}+U_{i+2}}{\Delta z^2}\right) \quad (11)$$

$$\frac{\partial^4 U}{\partial z^4}=\frac{6U_i+U_{i-2}-4U_{i-1}-4U_{i+1}+U_{i+2}}{\Delta z^4} \quad (12)$$

Substitute (11) and (12) to equation (3) at each discreted points, we can obtain a system of equations.

Boundary conditions

01-Impermeable top boundary at point 0

$$Lu_w[0]=0 \quad (13)$$

The condition of finctious node should be the same as a simply support beam. Singiresu (1980)

$$\frac{\partial^2 Lu_w[0]}{\partial z^2}=\frac{(Lu_w[1]-2Lu_w[0]+Lu_w[-1])}{\Delta z^2}=0 \quad (14)$$

Substitute (13) to (14) yeild

$$Lu_w[-1]=-Lu_w[1] \quad (15)$$

02-Applying stress of vacuum pressure at the top boundary

$$Lu_w[0]=L(P_{av}(t)) \quad (16)$$

where $L(P_{av}(t))$ is Laplace transform of applying stress or vaccum presssure.

The condition at finctious node should be the same condition of simply support beam.

$$\frac{\partial^2 Lu_w[0]}{\partial z^2}=\frac{(Lu_w[1]-2Lu_w[0]+Lu_w[-1])}{\Delta z^2}=0 \quad (17)$$

Substitute (16) to (17) to obtain

$$Lu_w[-1]=2L(P_{av})-Lu_w[1] \quad (18)$$

03-Impermeable bottom boundary at point N + 1

$$\frac{\partial Lu_w[N+1]}{\partial z}=\frac{(Lu_w[N+2]-Lu_w[N])}{2\Delta z}=0 \quad (19)$$

Therefore

$$Lu_w[N+2]=Lu_w[N] \quad (20)$$

The ending Laplace transform pore pressure should have the same value with previous value.

$$Lu_w[N+1]=Lu_w[N] \quad (21)$$

04-Permeable bottom boundary at point N + 1

$$\frac{\partial^2 Lu_w[N+1]}{\partial z^2}=\frac{(Lu_w[N+2]-2Lu_w[N+1]+Lu_w[N])}{\Delta z^2}=0 \quad (22)$$

Due to the assumption of permeable bottom therefore

$$Lu_w[N+1]=0 \quad (23)$$

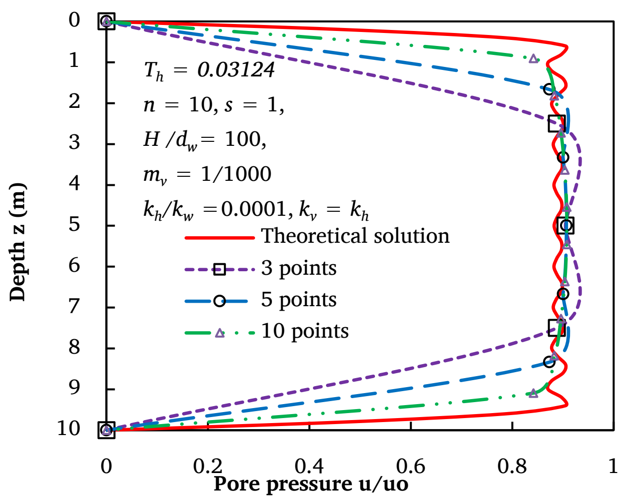
Substitute equation (23) to (21) yeild

$$Lu_w[N+2] = -Lu_w[N] \tag{24}$$

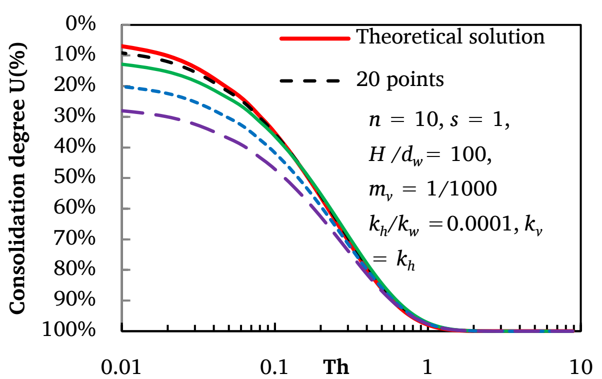
System of equations generate by substituting eqs (11) and (12) into (3) at each discreted points can be solved with boundary conditions (13), (15), (16), (18), (20), (21), (23) and (24) to obtain the Laplace tranform value of $Lu_w[i]$. $Lu_w[i]$ is then substitute into equation (4) to find the Laplace transform average pore pressure of soil $L\bar{u}_1(z,s)$. The average pore pressure at each point in soil will finally be determined by inverse Laplace technique $\bar{u}_1(z,t) = Inverse(L\bar{u}_1(z,s))$. There are a lot of numeric techniques for inversing Laplace stransform. This paper ultilizes the tool of Matlab, Maple or Mathematica softwares for simplifing inversed functions.

3. Verify present technique

3.1 Instant loading $\sigma(t) = u_o$



(a)



(b)

Figure 2. (a) comparison of pore pressure in case of instant loading (b) Comparison of average consolidation degree in case of instant loading.

This present technique is verified with theoretical consolidation solution of vertical drains by Carrillo's theoretical solution (1942) for combination of both vertical and horizontal drain. with intanst load, the present method analyses with 3, 5, 10 and 20 discreted points. The input parameters are given in the Figure 2

Figure 2(a) shows the comparison of pore pressure with three differences discreted points (3, 5 and 10 points) the more discreted points the more approximated estimation of pore pressure. The minimum discreted points (3 points) however achive a certain accurate at predicted points when compare with the theoritical solution. The largest error of pore pressure are in the top and bottom boundaries. Therefore, the present method excellent predicts pore pressure at discreted points but gains more error in the boundaries.

Figure 2(b) shows the comparison of average consolidation degree of four discreted points (3, 5, 10 and 20 points). The average consolidation degree by present technique is approximately derived by the sum of total area of pore pressure $\sum U_{pore}(t)$

$$\sum U_{pore}(t) = \sum_{i=0}^{i=N}\left(\frac{\bar{u}_1[z_i,t] + \bar{u}_1[z_{i+1},t]}{2}\Delta z\right) \tag{25}$$

The average consolidation degree is then found as following:

$$U(t) = 1 - \frac{\sum U_{pore}(t)}{Hu_o} \tag{26}$$

Figure (2b) clearly shows the average consolidation degree also depend on the number of discreted points. The maximum difference of consolidation degree between present solution and theoretical solution are 22.4 %, 14.1 %, 6.8 % and 2.6 % for cases 3 points, 5 points, 10 points and 20 points respectively. Finite difference method (FDM) with 20 points generates most acurated average consolidation degree however it required more computed time and efford. Each point content a whole information of pore pressure dissipation, load input and boundary condiction. Therefore it will push more computation cost for accuracy. Further study will be required to reduce the computation cost, more accuracy and less depend on special mathematic software to make this problem more applicable in common technical design.While FDM with 10 points can generate maximum difference less than 10 % (6.8 %) is good enough for predicting consolidation process and save more computed time.

3.2 Single ramp loading

The vertical drains with ramp load is widely applied in the soft ground treatment. Ramp load help reduce the plastic train and increase embankment stability. The present solution will analyse with 10 discreted points and the results are compared with the analytical solution by Tang and Onitsuka (2000)[14]. The final load is $q_u = 100$ (kPa), Time to complete loading step is $T_1 = 1$ and the input parameters are shown in the Figure 3.

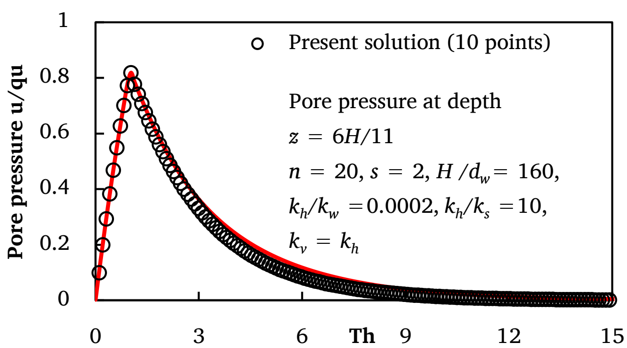
Figure 3(a) shows the comparison of pore pressure by the present solution and pore pressure by Tang and Onitsuka (2000)[14] solution at depth $z = 6H/11$ (point $i = 6$). Both solutions have almost match result. There are only slightly faster in dissipation of pore pressure by present solution at the nearly end stage of consolidation. However the final stage of consolidation both pore pressure curves meet again.

The comparison of average consolidation degree is shown in Figure. 3(b) the average consolidation degree by the present solution is determined by effective stress increment which transfer by pore pressure dissipation and is calculated as following:

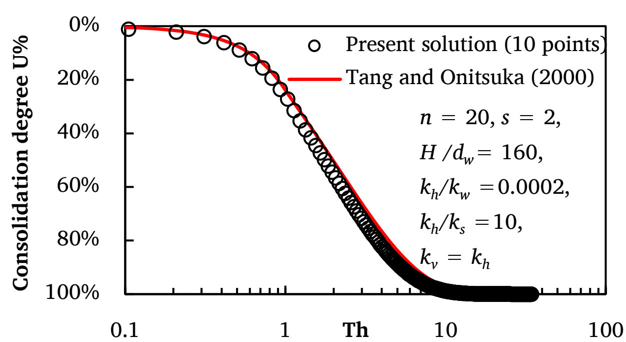
$$\sigma_i' = \sigma(t) - \bar{u}_1[z_i, t] \tag{27}$$

The total area of effective pressure increment is

$$\sum \sigma'(t) = \sum_{i=0}^{i=N} \left(\frac{\sigma_1'[z_i,t] + \sigma_1'[z_{i+1},t]}{2} \Delta z \right) \tag{28}$$



(a)



(b)

Figure 3. (a) comparison of pore pressure in case of single ramp loading

(b) Comparison of average consolidation degree in case of single ram loading.

The average consolidation degree is

$$U(t) = \frac{\sum \sigma'(t)}{Hq_u} \tag{29}$$

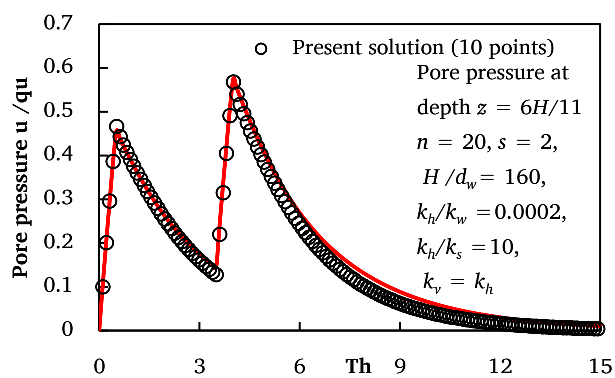
where q_u is the maximum surcharge load.

The average consolidation degree of present solution show slighly faster than Tang and Onitsuka's solution (2000)[14]. The maximum difference between two solution is 5.8 %.

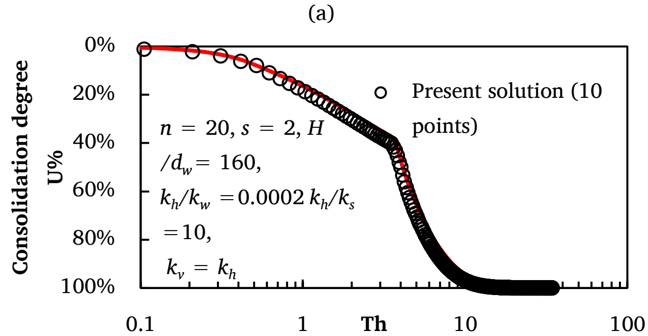
3.3 Multi-ramp loading

Multi-ramp loading for ground streatment with vertical drains is the most application on the site. With the same discreted 10 points, the two stage ramp loading are analyzed with present solution. The maximum applied load is $q_u = 100(kPa)$. The first loading step and second loading step with $T_1 = 0.5$, $T_2 = 3.5$, $T_3 = 4$, $q_1 = q_3 = 0.5q_u$, $q_2 = 0$. The results is then compared with the analytical solution of Tang and Onitsuka (2000)[14] (Figure 4).

The pore pressure dissipations at $z = 6H/11$ (point $i = 6$) are compared in Figure. 4a. both solutions have very match pore pressure estimations. Present solution have only slightly larger pore pressure at the latter stage of consolidation. The Fig 4b shows very good agreement of average degree of consolidation by present solution and Tang and Onitsuka's solution (2000)[14]. The maximum difference of degree of consolidation between two solutions is 4.9 % while there are 5.8 % and 6.8 % for single ramp loading and instant loading respectively. Therefore, FDM with discreted 10 points are more practical application for the real site with multi-ramp loading because it have less error in predicting average consolidation degree.



(a)



(b)

Figure 4. (a) comparison of pore pressure in case of multi-ramp loading

(b) Comparison of average consolidation degree in case of multi ram loading.

3.4 Application for real site construction by vacuum consolidation of prefarbricated vertical drains.

The present method is ultilized for analysis the ground treatment embankment with vertical drains and vacuum consolidation technique at Tianjin Port, China. Rujikiatkamjorn et al (2008)[16] investigated extensively the problem by FEM 2D and 3D. The analysis results were very good agreement with the field data. Geng at al (2012)[6] also analyzed the embankment with Laplace transform technique and inverse Laplace by numerical method of Durbin (1974)[4]. However the solution of Geng (2012)[6] is time comsumimg and very hard for estimate the multi-ramp loading case because Durbin's technique (1974)[4] is only suitable for predicting the smooth line of pore pressure (instant loading). Therefore, the present technique could efficiently apply for complex loading stages with complex boundary condition such as vacuum consolidation.

The present technique is applied with discreted 10 points. Loading of surcharge $\sigma(t)$ and Vacuum $P_{av}(t)$ have been declared as Figure 5(a). The scope of study of this paper is only considered in uniform soil therefore the soil parameters are converted into one equivalent uniform soil parameters. It is required further studies for multi soil layers in order to gain more accurate in estimating consolidation process. The equivalent vertical perneability is $k_v = 7.13\times10^{-10} m/s$, equivalent horizontal perneability $k_h = 22\times10^{-10} m/s$, The volumn compressive is assumed $m_v = 1/1500\, m^2/kN$ because of multi soil layers, smeared ratio $k_h/k_s = 2$, assuming no well resistance with $k_w = 100000k_h$. Other parameters are shown in Figure 5.

The pore pressures at location 5.5 m and 11 m depth are interpolated with the two nearest points. The pore pressure of these locations are compared with the results of Rujikiatkamjorn (2008)[16] by FEM 3D and field data (Figure 5(b) and 5(c)). The pore pressure by present method is the average pore pressure distribution in unit section, therefore it will be some limitations for comparison with the site data. However the present technique show very good agreement with the 3D FEM analysis by Rujikiatkamjorn (2008)[16]. Site date pore pressure show the same trend at location 11m depth while at location 5.5 m depth the site data of pore pressure at later consolidation stage show faster than present estimation.

The effective increase pressure will be determined by equation (27). This effective increase presssure latter ultilizes for estimating settlement by soil mechanics functions. The field date of settlements of surface, 3.8 m, 10.5 m and 14.5 m depth from surface are compared with present solutions. There are very good agreement at the surface, 3.8 m and 14.5 m results, however only location 10.5 m depth the field data shows faster in settlement at the latter consolidation stage.

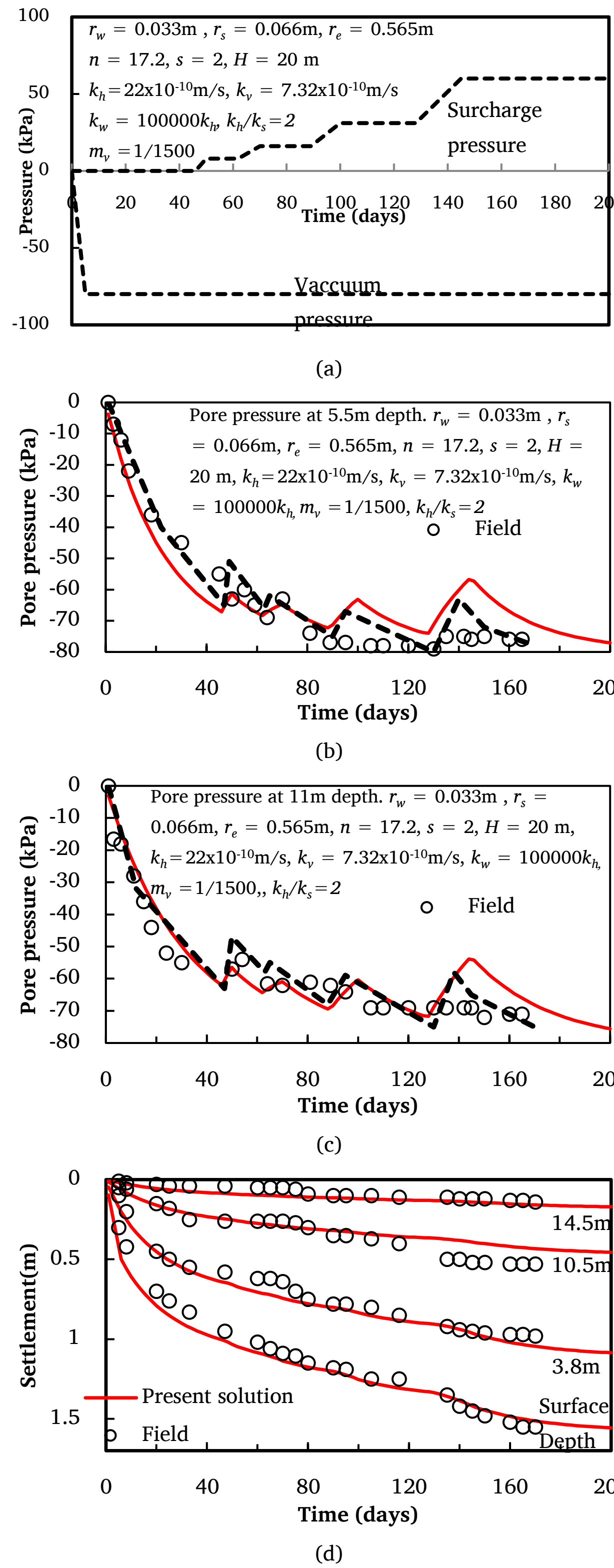


Figure 5. (a) Loading of Surcharge and Vacuum
(b) Comparison of pore pressure at 5.5 m depth
(c) Comparison of pore pressure at 11 m depth
(d) Comparison of settlement.

5-Summary and conclusion

Some conclusions can be drawn as follows:

1. This paper present the new finite different method with Laplace transform technique for vertical drains
2. This method can generate many load conditions and boundary conditions of instance load, ramp load, multi-ramp loads and even vacuum PVD.
3. The more loading steps of construction process the less error gains by present method. Therefore this method is very pratical
4. The more discreted points the more acurrate pore pressure achived when applying this technique. But the more computer time and effort requirement. This paper suggest a 10 point divide in estimating consolidation process.

Acknowledgements

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