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Investigation of different heaviside function forms in discontinuity problems

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KEYWORDS

Heaviside function
Discontinuity problems
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Numerical solution
Enrichment functions

ABSTRACT

This study investigates the impact of different mathematical forms of the Heaviside function on the numerical solution of discontinuity problems using the extended finite element method (XFEM). The Heaviside function is commonly employed to represent displacement jumps across discontinuities, such as cracks or material interfaces, through enrichment functions. Although several formulations exist, it remains unclear whether these differences influence the accuracy or consistency of numerical results. In this work, multiple cases involving a one-dimensional bar with an internal discontinuity are analyzed using distinct Heaviside formulations. Results show that, despite variations in the enrichment terms, the computed nodal displacements and overall structural responses are identical across all cases. This confirms that the numerical solution is invariant with respect to the specific form of the Heaviside function used. The findings support the robustness of XFEM and offer practical flexibility for implementation in discontinuity modeling without compromising numerical reliability.

1. Introduction

The Heaviside function is one of the fundamental enrichment functions employed in the extended finite element method (XFEM) to model strong discontinuities, such as cracks, interfaces, and material separations. In XFEM, the standard finite element approximation is enriched by discontinuous functions to capture jumps in the displacement field without modifying the mesh. This approach has significantly improved the numerical simulation of fracture and discontinuity problems, offering advantages in accuracy and computational efficiency [1–4].

Since the introduction of XFEM by Moës et al. [5], the Heaviside function has been widely used to model open cracks and displacement discontinuities. It has become a standard tool in linear elastic fracture mechanics [6–7], cohesive zone models [8], and crack growth simulations in two and three dimensions [9–10]. However, despite its popularity, the Heaviside function does not have a universally accepted definition. Various formulations exist in the literature, which differ in how values are assigned to the two sides of a discontinuity.

Three main forms of the Heaviside function are commonly encountered: The symmetric form, $H(x) = -1$ for $x < x_c$ and $H(x) = +1$ for $x > x_c$, has been adopted in classical works such as Zhao, et al. [11]. The shifted form, $H(x) = 0$ for $x < x_c$ and $H(x) = +1$ for $x > x_c$, is often used for numerical convenience, appearing in studies by Hansbo and Hansbo [12], Wu [13], and Lang et al. [14,15], as well as in Dolbow and Devan [16]. The reversed form, $H(x) = +1$ for $x < x_c$ and $H(x) = -1$ for $x > x_c$, appears in works following alternative sign conventions, such as Moës et al. [5], Sukumar et al. [17], and Jiang et al. [18].

Although all three forms are mathematically valid, they may influence the structure of the enriched shape functions and, consequently, the stiffness matrix and numerical solution. Yet, only limited efforts have been made to systematically evaluate whether these differences affect the computed structural response. Most studies adopt one form and assume that the numerical results are independent of the specific definition, without rigorous verification.

This paper addresses this gap by investigating the influence of different Heaviside function forms on XFEM solutions through a one-dimensional benchmark problem. By comparing displacement fields and enriched degrees of freedom across the three definitions under identical conditions, the study aims to clarify whether the XFEM formulation exhibits solution invariance. The findings contribute to a deeper understanding of XFEM enrichment and offer practical guidelines for its implementation in discontinuity problems.

2. Basic theory of enriched functions

In the extended finite element method (XFEM), the displacement field is enhanced through enrichment functions to represent discontinuities without modifying the mesh. For problems involving strong discontinuities such as open cracks, the standard displacement approximation is enriched with a Heaviside function to allow for discontinuous jumps across the crack. In this section, we calculate the displacement at each node of the bar for the previously defined cases, and then compare the results to evaluate any differences.

Steps to compute the nodal displacements:

- Define the stiffness matrix for each element;
- Assemble the local stiffness matrices into a global stiffness matrix;
- Use the main XFEM equation [5] to find the nodal displacements u:

$$\boldsymbol{Ku} = \boldsymbol{F} \quad (1)$$

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$$\text{Where } \boldsymbol{K} = \begin{bmatrix} \boldsymbol{K_{uu}} & \boldsymbol{K_{ua}} \\ \boldsymbol{K_{au}} & \boldsymbol{K_{aa}} \end{bmatrix} \quad (2)$$

K_{uu} represents stiffness matrix of the standard (non-enriched) element:

$$\boldsymbol{K_{uu}} = \int_0^l (\boldsymbol{B}^u_{std})^T \boldsymbol{D} \boldsymbol{B}^u_{std}\, dx \quad (3)$$

K_{ua} or K_{au} represent coupling matrices between standard and enriched degrees of freedom:

$$\boldsymbol{K}_{ua} = \int_0^l (\boldsymbol{B}^u_{std})^T \boldsymbol{D} \boldsymbol{B}^a_{enr} dx,\ \boldsymbol{K}_{au} = \int_0^l (\boldsymbol{B}^a_{enr})^T \boldsymbol{D} \boldsymbol{B}^u_{std} dx \quad (4)$$

K_{aa} represents stiffness matrix of the enriched element:

$$\boldsymbol{K}_{aa} = \int_0^l (\boldsymbol{B}^a_{enr})^T \boldsymbol{D} \boldsymbol{B}^a_{enr} dx \quad (5)$$

3. Examples and discussions

Consider a model as Figure 1, a one-dimensional bar subjected to axial tension, with a discontinuity located between two interior nodes. The domain is discretized into three equal-length finite elements, each of length L, resulting in four nodes x_1 to x_4. The bar is fixed at the left end and subjected to a prescribed displacement ū at the right end. A displacement discontinuity is introduced between nodes x_2 and x_3, representing a crack or interface where enriched degrees of freedom are applied. This simplified model serves as a benchmark problem for assessing the influence of different Heaviside function forms on the XFEM solution.

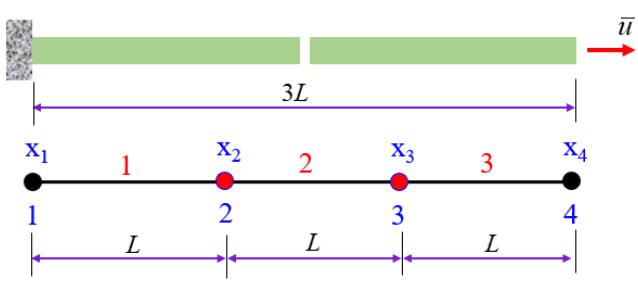


Figure 1. The one-dimensional model with discontinuity under axial tension.

3.1. Symmetric Heaviside function $H(x) = \{-1, +1\}$

In this case, the Heaviside function is defined as follows:

$$H(x) = \begin{cases} -1, & x < 0 \\ +1, & x \geq 0 \end{cases} \quad (6)$$

Figure 2 depicts the enriched finite element model using the symmetric Heaviside function, defined such that $H(x) = -1$ for $x < x_c$ and $H(x) = +1$ for $x \geq x_c$. The discontinuity is introduced at the midpoint of element 2, between nodes 2 and 3. Nodal displacements are denoted as u_1 to u_4. Enriched degrees of freedom a_1 and a_2 are applied at nodes 2 and 3, respectively, corresponding to the Heaviside values $H(x_2) = -1$ and $H(x_3) = +1$. On the right-hand side, the enriched shape functions $N_1(x)H(x)$ and $N_2(x)H(x)$ are shown, exhibiting a sign change across the discontinuity. This symmetric form introduces an antisymmetric enrichment around the crack.

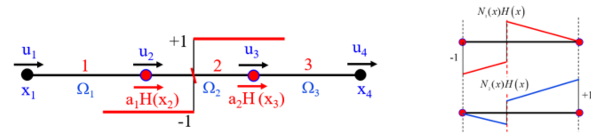


Figure 2. Enriched finite element model of a 1D with a displacement discontinuity represented by $H(x) = -1$ and $H(x) = +1$.

The stiffness of element 1: we add the Heaviside function at node 2 with $H(x) = -1$.

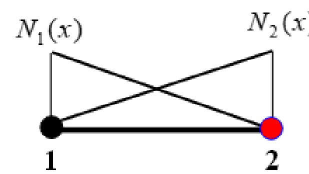


Figure 3. Shape functions for nodes 1 and 2 of element 1.

For 1D linear element we have shape function at node 1 and 2 as the following

$$\boldsymbol{N_1(x) = 1 - \frac{x}{L}}$$

$$\boldsymbol{N_2(x)} = \frac{x}{L} \quad (7)$$

$$\boldsymbol{B_1} = [\boldsymbol{B^u_{std}} \quad \boldsymbol{B^a_{enr}}] = \begin{bmatrix} \frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_2(x))}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & -\frac{1}{L} \end{bmatrix} \ (8)$$

$$\text{Where } \frac{\partial (HN_2(x))}{\partial x} = \boldsymbol{H} \frac{\partial (N_2(x))}{\partial x} = \boldsymbol{H} \frac{\partial \left(\frac{x}{L}\right)}{\partial x} = \boldsymbol{H}\frac{1}{L} = -\boldsymbol{1}\frac{1}{L} = -\frac{1}{L} \ (9)$$

$$\boldsymbol{K_1} = \int_0^L (\boldsymbol{B_1})^T \boldsymbol{EAB_1 dx} = \boldsymbol{EAL} \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ -\frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & -\frac{1}{L} \end{bmatrix} = \frac{EA}{L} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & a_1 \end{array} \\ \begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{1} \\ \boldsymbol{-1} & \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{1} \end{bmatrix} \end{array} \begin{array}{c} \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{a_1} \end{array} \quad (10)$$

The stiffness of element 2:

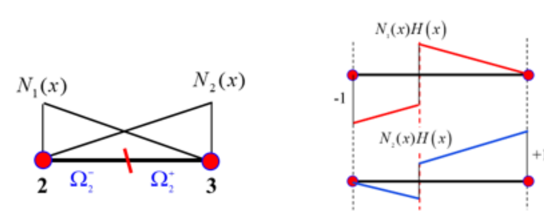


Figure 4. Shape functions for nodes 2 and 3 of element 2.

At element 2 it has a crack at the middle so the length of the element 2 discontinuity, therefore we must divide by 2 parts to calculate the stiffness matrix of this element. The left-hand side of the crack add the Heaviside function at node 2 $H(x) = -1$ and the right-hand side of node 3 $H(x) = +1$.

The stiffness of element 2 is calculated as the following $K_2 = K_2^- + K_2^+$

$$\boldsymbol{B_2^-} = [\boldsymbol{B^u_{std}} \quad \boldsymbol{B^a_{enr}}] = \begin{bmatrix} \frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_1(x))}{\partial x} & \frac{\partial (HN_2(x))}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & \frac{1}{L} & -\frac{1}{L} \end{bmatrix} \quad (11)$$

$$\boldsymbol{K}_2^- = \int_0^{L/2} (\boldsymbol{B}_2^-)^T EA\boldsymbol{B}_2^- dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ \frac{1}{L} \\ -\frac{1}{L} \end{bmatrix} \begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} & \frac{1}{L} & \frac{-1}{L} \end{bmatrix} \end{array} =$$

$$\frac{EA}{L} \begin{bmatrix} 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \\ -0{,}5 & 0{,}5 & 0{,}5 & -0{,}5 \\ -0{,}5 & 0{,}5 & 0{,}5 & -0{,}5 \\ 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \\ a_1 \\ a_2 \end{array} \quad (12)$$

$$\boldsymbol{B}_2^+ = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \begin{bmatrix} \frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_1(x))}{\partial x} & \frac{\partial (HN_2(x))}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \quad (13)$$

$$\boldsymbol{K}_2^+ = \int_0^{L/2} (\boldsymbol{B}_2^+)^T EA\boldsymbol{B}_2^+ dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} & -\frac{1}{L} & \frac{1}{L} \end{bmatrix} =$$

$$\frac{EA}{L} \begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 0{,}5 & -0{,}5 & 0{,}5 & -0{,}5 \\ -0{,}5 & 0{,}5 & -0{,}5 & 0{,}5 \\ 0{,}5 & -0{,}5 & 0{,}5 & -0{,}5 \\ -0{,}5 & 0{,}5 & -0{,}5 & 0{,}5 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_3 \\ a_1 \\ a_2 \end{array} \quad (14)$$

$$\Rightarrow \boldsymbol{K}_2 = \boldsymbol{K}_2^- + \boldsymbol{K}_2^+ = \frac{EA}{L} \begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_2 \\ a_1 \\ a_2 \end{array} \quad (15)$$

- The stiffness of element 3: we add the Heaviside function at node 3 with $H(x) = +1$.

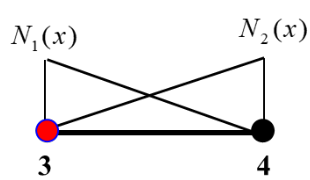


Figure 5. Shape functions for nodes 3 and 4 of element 3.

$$\boldsymbol{B}_3 = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \begin{bmatrix} \frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_1(x))}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & \frac{1}{L} \end{bmatrix} \quad (16)$$

$$\boldsymbol{K}_3 = \int_0^{L} (\boldsymbol{B}_3)^T EA\boldsymbol{B}_3 dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & \frac{1}{L} \end{bmatrix} = \frac{EA}{L} \begin{array}{c} \begin{array}{ccc} u_3 & u_4 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_3 \\ u_4 \\ a_2 \end{array} \quad (17)$$

Now we assemble for global matrix K in this case

$$\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2 + \boldsymbol{K}_3 = \frac{EA}{L} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2 \end{array} \quad (18)$$

This is control displacement problem so we have

$$F_j = \boldsymbol{0} \Leftrightarrow \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_{a1} \\ f_{a2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (19)$$

According to boundary condition in Figure 1, we have

$$u_1 = 0$$

$$u_4 = \bar{u} \quad (20)$$

Using governing equation $K_{ij}u_j = F_j$ and boundary condition, we can write as

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ a_1 \\ a_2 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} 0 \\ \bar{u} \\ 0 \\ \bar{u} \end{bmatrix} \Rightarrow \begin{cases} u_2 = \bar{u}/2 \\ u_3 = \bar{u}/2 \\ a_1 = \bar{u}/2 \\ a_2 = \bar{u}/2 \end{cases} \quad (21)$$

Applying interpolation in XFEM

$$u(x) = N_i u_i + HN_j a_j$$
$$N_i = N_{std}^u$$
$$HN_j = N_{enr}^a \quad (22)$$

We can get the true displacement at each node as

$$u(x_1) = u_1 = 0$$
$$u(x_2) = u_2 + H(x_2)a_1 = \frac{\bar{u}}{2} + (-1)\left(\frac{\bar{u}}{2}\right) = 0$$
$$u(x_3) = u_3 + H(x_3)a_3 = \frac{\bar{u}}{2} + (+1)\left(\frac{\bar{u}}{2}\right) = \bar{u}$$
$$u(x_4) = \bar{u} \quad (23)$$

This result matches the exact solution: $u_1(x_1) = 0, u_2(x_2) = 0, u_3(x_3) = \bar{u}, u_4(x_4) = \bar{u}$

3.2. *Shifted Heaviside function $H(x) = \{0, +1\}$*

In this case, the Heaviside function is defined such that the value is zero on the left-hand side of the discontinuity and one on the right-hand side, as follows:

$$H(x) = \begin{cases} 0, & x < 0 \\ +1, & x \geq 0 \end{cases} \quad (24)$$

Figure 6 illustrates the enriched finite element model for the case where the shifted Heaviside function is used, defined as $H(x) = 0$ for $x < x_c$ and $H(x) = +1$ for $x \geq x_c$. The displacement discontinuity is located between nodes 2 and 3, where enriched degrees of freedom a_1 and a_2 are introduced. Since $H(x_2) = 0$, no enrichment is applied at node 2, while $H(x_3) = +1$ activates the enrichment at node 3. The right side of the figure shows the enriched shape functions $N_1(x)H(x)$ and $N_2(x)H(x)$, which exhibit a displacement jump across the interface. This formulation allows the discontinuity to be captured asymmetrically, making it suitable for modeling problems with unidirectional displacement jumps.

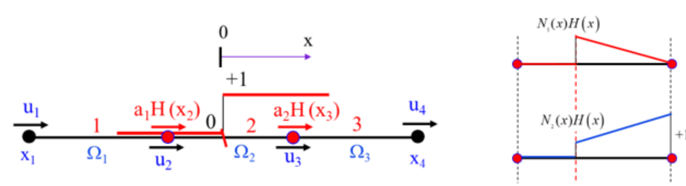


Figure 6. Enriched finite element model of a 1D with a displacement discontinuity represented by $H(x) = 0$ and $H(x) = +1$.

The stiffness of element 1: we add the Heaviside function at node 2 with $H(x) = 0$.

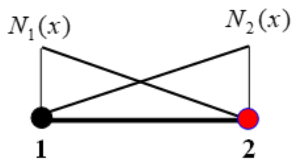


Figure 7. Shape functions for nodes 1 and 2 of element 1.

$$\boldsymbol{B}_1 = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \begin{bmatrix}\frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_2(x))}{\partial x}\end{bmatrix} = \begin{bmatrix}-\frac{1}{L} & \frac{1}{L} & 0\end{bmatrix} \tag{25}$$

$$\boldsymbol{K}_1 = \int_0^L (\boldsymbol{B}_1)^T EA\boldsymbol{B}_1 dx = EAL\begin{bmatrix}\frac{-1}{L}\\ \frac{1}{L}\\ 0\end{bmatrix}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L} & 0\end{bmatrix} = \frac{EA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{matrix}u_1\\ u_2\\ a_1\end{matrix} \tag{26}$$

The stiffness of element 2: at node 2 with $H(x) = 0$ and at node 3 with $H(x) = 1$.

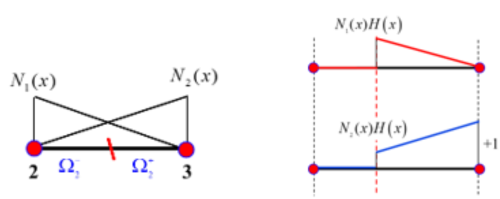


Figure 8. Shape functions for nodes 2 and 3 of element 2.

At element 2 it has a crack at the middle so the length of the element 2 discontinuity, therefore we must divide by 2 parts to calculate the stiffness matrix of this element. The left-hand side of the crack add the Heaviside function at node 2 $H(x) = 0$ and the right-hand side of node 3 $H(x) = +1$.

The stiffness of element 2 is calculated as the following $K_2 = K_2^- + K_2^+$

$$\boldsymbol{B}_2^- = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \begin{bmatrix}\frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_1(x))}{\partial x} & \frac{\partial (HN_2(x))}{\partial x}\end{bmatrix} = \begin{bmatrix}-\frac{1}{L} & \frac{1}{L} & 0 & 0\end{bmatrix} \tag{27}$$

$$\boldsymbol{K}_2^- = \int_0^{L/2} (\boldsymbol{B}_2^-)^T EA\boldsymbol{B}_2^- dx = EAL\begin{bmatrix}\frac{-1}{L}\\ \frac{1}{L}\\ 0\\ 0\end{bmatrix}\begin{bmatrix}\frac{-1}{L} & \frac{1}{L} & 0 & 0\end{bmatrix} =$$

$$\begin{array}{cc} & \begin{matrix} u_2 & u_3 & a_1 & a_2 \end{matrix}\\ \frac{EA}{L} & \begin{bmatrix}0{,}5 & -0{,}5 & 0 & 0\\ -0{,}5 & 0{,}5 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{matrix}u_2\\ u_2\\ a_1\\ a_2\end{matrix}\end{array} \tag{28}$$

$$\boldsymbol{B}_2^+ = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \begin{bmatrix}\frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_1(x))}{\partial x} & \frac{\partial (HN_2(x))}{\partial x}\end{bmatrix} = \begin{bmatrix}-\frac{1}{L} & \frac{1}{L} & -\frac{1}{L} & \frac{1}{L}\end{bmatrix} \tag{29}$$

$$\boldsymbol{K}_2^+ = \int_0^{L/2} (\boldsymbol{B}_2^+)^T EA\boldsymbol{B}_2^+ dx = EAL\begin{bmatrix}\frac{-1}{L}\\ \frac{1}{L}\\ -\frac{1}{L}\\ \frac{1}{L}\end{bmatrix}\begin{bmatrix}\frac{-1}{L} & \frac{1}{L} & -\frac{1}{L} & \frac{1}{L}\end{bmatrix} =$$

$$\begin{array}{cc} & \begin{matrix} u_2 & u_3 & a_1 & a_2 \end{matrix}\\ \frac{EA}{L} & \begin{bmatrix}0{,}5 & -0{,}5 & 0{,}5 & -0{,}5\\ -0{,}5 & 0{,}5 & -0{,}5 & 0{,}5\\ 0{,}5 & -0{,}5 & 0{,}5 & -0{,}5\\ -0{,}5 & 0{,}5 & -0{,}5 & 0{,}5\end{bmatrix}\begin{matrix}u_2\\ u_2\\ a_1\\ a_2\end{matrix}\end{array} \tag{30}$$

$$\Rightarrow \boldsymbol{K}_2 = \boldsymbol{K}_2^- + \boldsymbol{K}_2^+ = \begin{array}{cc} & \begin{matrix} u_2 & u_3 & a_1 & a_2 \end{matrix}\\ \frac{EA}{L} & \begin{bmatrix}1 & -1 & 0{,}5 & -0{,}5\\ -1 & 1 & -0{,}5 & 0{,}5\\ 0{,}5 & -0{,}5 & 1 & -1\\ -0{,}5 & 0{,}5 & -1 & 1\end{bmatrix}\begin{matrix}u_2\\ u_2\\ a_1\\ a_2\end{matrix}\end{array} \tag{31}$$

The stiffness of element 3: we add the Heaviside function at node 3 with $H(x) = +1$.

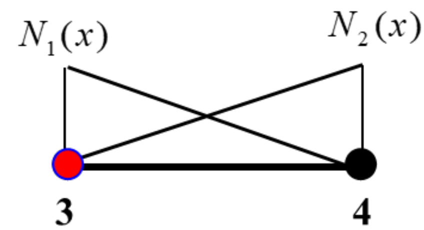


Figure 9. Shape functions for nodes 3 and 4 of element 3.

$$\boldsymbol{B}_3 = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \begin{bmatrix}\frac{\partial N_1(x)}{\partial x} & \frac{\partial N_2(x)}{\partial x} & \frac{\partial (HN_1(x))}{\partial x}\end{bmatrix} = \begin{array}{c}\begin{matrix} u_1 & u_2 & a_1 \end{matrix}\\ \begin{bmatrix}-\frac{1}{L} & \frac{1}{L} & \frac{1}{L}\end{bmatrix}\end{array} \tag{32}$$

$$\boldsymbol{K}_3 = \int_0^L (\boldsymbol{B}_3)^T EA\boldsymbol{B}_3 dx = EAL\begin{bmatrix}\frac{-1}{L}\\ \frac{1}{L}\\ \frac{1}{L}\end{bmatrix}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L} & \frac{1}{L}\end{bmatrix} = \begin{array}{cc} & \begin{matrix} u_3 & u_4 & a_2 \end{matrix}\\ \frac{EA}{L} & \begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{matrix}u_3\\ u_4\\ a_2\end{matrix}\end{array} \tag{33}$$

Now we assemble for global matrix K in this case

$$\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2 + \boldsymbol{K}_3 = \begin{array}{cc} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & a_1 & a_2 \end{matrix}\\ \frac{EA}{L} & \begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0{,}5 & -0{,}5\\ 0 & -1 & 2 & -1 & -0{,}5 & 1{,}5\\ 0 & 0 & -1 & 1 & 0 & -1\\ 0 & 0{,}5 & -0{,}5 & 0 & 0{,}5 & -0{,}5\\ 0 & -0{,}5 & 1{,}5 & -1 & -0{,}5 & 1{,}5\end{bmatrix}\begin{matrix}u_1\\ u_2\\ u_3\\ u_4\\ a_1\\ a_2\end{matrix}\end{array} \tag{34}$$

This is control displacement problem so we have

$$F_j = \mathbf{0} \Leftrightarrow \begin{Bmatrix}f_1\\ f_2\\ f_3\\ f_4\\ f_{a1}\\ f_{a2}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{Bmatrix} \tag{35}$$

According to boundary condition in Figure 1, we have

$$u_1 = 0$$
$$u_4 = \bar{u} \tag{36}$$

Using governing equation $K_{ij}u_j = F_j$ and boundary condition, we can write as

$$\begin{bmatrix}2 & -1 & 0{,}5 & -0{,}5\\ -1 & 2 & -0{,}5 & 1{,}5\\ 0{,}5 & -0{,}5 & 0{,}5 & -0{,}5\\ -0{,}5 & 1{,}5 & -0{,}5 & 1{,}5\end{bmatrix}\begin{bmatrix}u_2\\ u_3\\ a_1\\ a_2\end{bmatrix} = \frac{L}{EA}\begin{bmatrix}0\\ \bar{u}\\ 0\\ \bar{u}\end{bmatrix} \Rightarrow \begin{cases}u_2 = -2{,}22\times 10^{-16}\bar{u}\\ u_3 = -8{,}88\times 10^{-16}\bar{u}\\ a_1 = \bar{u}\\ a_2 = \bar{u}\end{cases} \tag{37}$$

Applying interpolation in XFEM

$$u(x) = N_i u_i + HN_j a_j$$
$$N_i = N_{std}^u$$
$$HN_j = N_{enr}^a \tag{38}$$

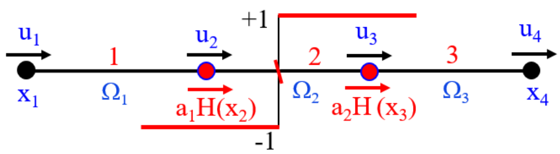
We can get the true displacement at each node as

$$u(x_1) = u_1 = 0$$
$$u(x_2) = u_2 + H(x_2)a_1 = -2.22\times 10^{-16}\bar{u} + (0)(\bar{u}) = -2.22\times 10^{-16}\bar{u} \approx 0$$
$$u(x_3) = u_3 + H(x_3)a_3 = -8.88\times 10^{-16}\bar{u} + (+1)\bar{u} \approx \bar{u}$$
$$u(x_4) = \bar{u} \tag{39}$$

This result matches the exact solitio$u_1(x_1) = 0, u_2(x_2) = 0, u_3(x_3) = \bar{u}, u_4(x_4) = \bar{u}$

Compare the result between case 1 and case 2 as follows

Case 1



$$u_1 = 0$$
$$u_2 = \bar{u}/2$$
$$u_3 = \bar{u}/2$$
$$u_4 = \bar{u}$$
$$a_1 = \bar{u}/2$$

$$a_2 = \bar{u}/2 \Rightarrow \begin{cases} u(x_1) = u_1 = 0 \\ u(x_2) = \frac{\bar{u}}{2} + (-1)\left(\frac{\bar{u}}{2}\right) = 0 \\ u(x_3) = \frac{\bar{u}}{2} + (+1)\left(\frac{\bar{u}}{2}\right) = \bar{u} \\ u(x_4) = \bar{u} \end{cases}$$

Comments: By comparing the results of Case 1 and Case 2, it is clear that although the enriched degrees of freedom and intermediate expressions differ, the final computed nodal displacements at all points are numerically identical. In Case 1, the symmetric Heaviside function leads to enriched coefficients $a_1 = a_2 = \bar{u}/2$, while in Case 2, the shifted form results in $a_1 = a_2 = \bar{u}$, yet both yield the same physical displacements at nodes x_1, x_2, x_3, and x_4. This consistency confirms that the choice of Heaviside function form–whether symmetric or shifted–does not affect the overall displacement solution in XFEM for this benchmark problem. Therefore, the formulation exhibits solution invariance, and either form can be reliably used to model displacement discontinuities.

3.3. *Reversed Heaviside function* $H(x) = \{+1, -1\}$

In this case, the Heaviside function is defined such that it takes a positive value on the left-hand side of the discontinuity and a negative value on the right-hand side. This reversed definition is expressed mathematically as follows:

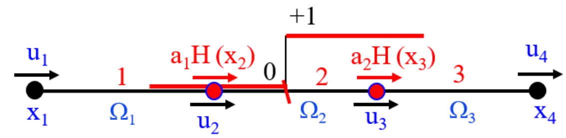
$$H(x) = \begin{cases} +1, & x < 0 \\ -1, & x \geq 0 \end{cases} \quad (40)$$

Figure 10 illustrates a one-dimensional finite element model enriched with a Heaviside function to represent a displacement discontinuity. The domain is divided into three elements, with a crack or interface located between nodes 2 and 3. Nodal displacements are denoted by u_1 to u_4. Enriched degrees of freedom a_1 and a_2 are introduced at nodes 2 and 3 through the functions $H(x_2)$ and $H(x_3)$, respectively. The right side of the figure shows the shape functions $N_1(x)H(x)$ and $N_2(x)H(x)$, which are discontinuous across the interface, demonstrating the effect of the Heaviside function where $H(x) = +1$ on the left and $H(x) = -1$ on the right.

The stiffness of element 1: we add the Heaviside function at node 2 with $H(x) = +1$.

The stiffness of element 2: at node 2 with $H(x) = +1$ and at node 3 with $H(x) = -1$.

Case 2



$$u_1 = 0$$
$$u_2 = -2{,}22 \times 10^{-16}\bar{u}$$
$$u_3 = -8{,}88 \times 10^{-16}\bar{u}$$
$$u_4 = \bar{u}$$
$$a_1 = \bar{u}$$

$$a_2 = \bar{u} \Rightarrow \begin{cases} u(x_1) = u_1 = 0 \\ u(x_2) = -2{,}22 \times 10^{-16}\bar{u} + (0)(\bar{u}) \approx 0 \\ u(x_3) = -8{,}88 \times 10^{-16}\bar{u} + (+1)\bar{u} \approx \bar{u} \\ u(x_4) = \bar{u} \end{cases}$$

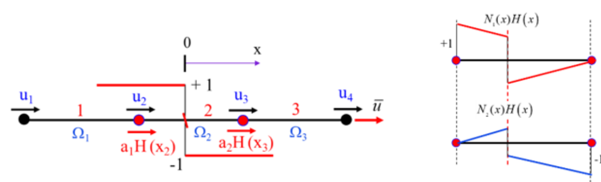


Figure 10. Enriched finite element model of a 1D with a displacement discontinuity represented by $H(x) = +1$ and $H(x) = -1$.

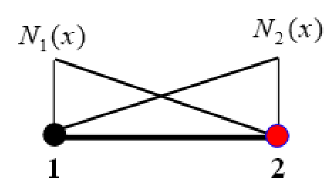


Figure 11. Shape functions for nodes 1 and 2 of element 1.

$$\boldsymbol{B}_1 = [\boldsymbol{B}^u_{std} \quad \boldsymbol{B}^a_{enr}] = \left[\frac{\partial N_1(x)}{\partial x} \quad \frac{\partial N_2(x)}{\partial x} \quad \frac{\partial (HN_2(x))}{\partial x}\right] = \left[-\frac{1}{L} \quad \frac{1}{L} \quad \frac{1}{L}\right] \quad (41)$$

$$\boldsymbol{K}_1 = \int_0^L (\boldsymbol{B}_1)^T EA\boldsymbol{B}_1 dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \left[-\frac{1}{L} \quad \frac{1}{L} \quad \frac{1}{L}\right] = \frac{EA}{L} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & a_1 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ a_1 \end{array} \quad (42)$$

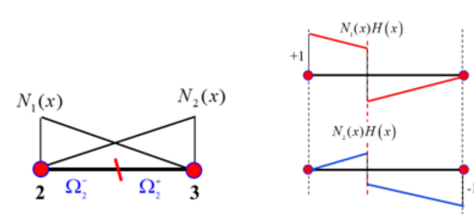


Figure 12. Shape functions for nodes 2 and 3 of element 2.

At element 2 it has a crack at the middle so the length of the element 2 discontinuity, therefore we must divide by 2 parts to calculate the stiffness matrix of this element. The left-hand side of the crack add the Heaviside function at node 2 $H(x) = +1$ and the right-hand side of node 3 $H(x) = -1$.

The stiffness of element 2 is calculated as the following $K_2 = K_2^- + K_2^+$

$$\boldsymbol{B}_2^- = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \left[\frac{\partial N_1(x)}{\partial x} \quad \frac{\partial N_2(x)}{\partial x} \quad \frac{\partial (HN_1(x))}{\partial x} \quad \frac{\partial (HN_2(x))}{\partial x}\right] = \left[-\frac{1}{L} \quad \frac{1}{L} \quad -\frac{1}{L} \quad \frac{1}{L}\right] \quad (43)$$

$$\boldsymbol{K}_2^- = \int_0^{\frac{L}{2}} (\boldsymbol{B}_2^-)^T EA\boldsymbol{B}_2^- dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} & \frac{-1}{L} & \frac{1}{L} \end{bmatrix} =$$

$$\frac{EA}{L} \begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \\ -0{,}5 & 0{,}5 & 0{,}5 & -0{,}5 \\ 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \\ -0{,}5 & 0{,}5 & 0{,}5 & -0{,}5 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_2 \\ a_1 \\ a_2 \end{array} \quad (44)$$

$$\boldsymbol{B}_2^+ = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \left[\frac{\partial N_1(x)}{\partial x} \quad \frac{\partial N_2(x)}{\partial x} \quad \frac{\partial (HN_1(x))}{\partial x} \quad \frac{\partial (HN_2(x))}{\partial x}\right] = \left[-\frac{1}{L} \quad \frac{1}{L} \quad \frac{1}{L} \quad \frac{-1}{L}\right] \quad (45)$$

$$\boldsymbol{K}_2^+ = \int_0^{\frac{L}{2}} (\boldsymbol{B}_2^+)^T EA\boldsymbol{B}_2^+ dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ \frac{1}{L} \\ \frac{-1}{L} \end{bmatrix} \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} & \frac{1}{L} & \frac{-1}{L} \end{bmatrix} =$$

$$\frac{EA}{L} \begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 0{,}5 & -0{,}5 & 0{,}5 & -0{,}5 \\ -0{,}5 & 0{,}5 & -0{,}5 & 0{,}5 \\ -0{,}5 & 0{,}5 & -0{,}5 & 0{,}5 \\ 0{,}5 & -0{,}5 & 0{,}5 & -0{,}5 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_2 \\ a_1 \\ a_2 \end{array} \quad (46)$$

$$\Rightarrow \boldsymbol{K}_2 = \boldsymbol{K}_2^- + \boldsymbol{K}_2^+ = \frac{EA}{L} \begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_2 \\ a_1 \\ a_2 \end{array} \quad (47)$$

The stiffness of element 3: we add the Heaviside function at node 3 with $H(x) = -1$.

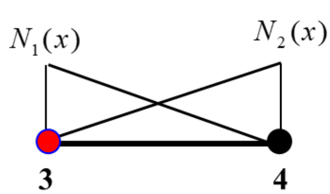
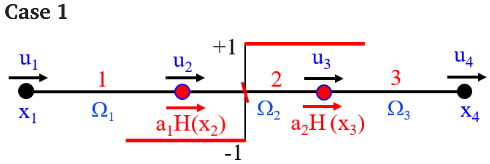


Figure 13. Shape functions for nodes 3 and 4 of element 3.

Case 1



$$u_1 = 0$$
$$u_2 = \bar{u}/2$$
$$u_3 = \bar{u}/2$$
$$u_4 = \bar{u}$$
$$a_1 = \bar{u}/2$$

$$a_2 = \bar{u}/2 \Rightarrow \begin{cases} u(x_1) = u_1 = 0 \\ u(x_2) = \frac{\bar{u}}{2} + (-1)\left(\frac{\bar{u}}{2}\right) = 0 \\ u(x_3) = \frac{\bar{u}}{2} + (+1)\left(\frac{\bar{u}}{2}\right) = \bar{u} \\ u(x_4) = \bar{u} \end{cases}$$

$$\boldsymbol{B}_3 = [\boldsymbol{B}_{std}^u \quad \boldsymbol{B}_{enr}^a] = \left[\frac{\partial N_1(x)}{\partial x} \quad \frac{\partial N_2(x)}{\partial x} \quad \frac{\partial (HN_1(x))}{\partial x}\right] = \left[-\frac{1}{L} \quad \frac{1}{L} \quad \frac{1}{L}\right] \quad (48)$$

$$\boldsymbol{K}_3 = \int_0^L (\boldsymbol{B}_3)^T EA\boldsymbol{B}_3 dx = EAL \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & \frac{1}{L} \end{bmatrix} = \frac{EA}{L} \begin{array}{c} \begin{array}{ccc} u_3 & u_4 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_3 \\ u_4 \\ a_2 \end{array} \quad (49)$$

Now we assemble for global matrix K in this case

$$\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2 + \boldsymbol{K}_3 = \frac{EA}{L} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 & -1 & 2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2 \end{array} \quad (50)$$

This is control displacement problem so we have

$$F_j = \boldsymbol{0} \Leftrightarrow \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_{a1} \\ f_{a2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (51)$$

According to boundary condition in Figure 1, we have

$$u_1 = 0$$
$$u_4 = \bar{u} \quad (52)$$

Using governing equation $K_{ij}u_j = F_j$ and boundary condition, we can write as

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ a_1 \\ a_2 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} 0 \\ \bar{u} \\ 0 \\ -\bar{u} \end{bmatrix} \Rightarrow \begin{cases} u_2 = \bar{u}/2 \\ u_3 = \bar{u}/2 \\ a_1 = -\bar{u}/2 \\ a_2 = -\bar{u}/2 \end{cases} \quad (53)$$

Applying interpolation in XFEM

$$u(x) = N_i u_i + HN_j a_j$$
$$N_i = N_{std}^u$$
$$HN_j = N_{enr}^a \quad (54)$$

We can get the true displacement at each node as

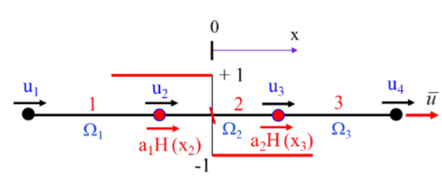
$$u(x_1) = u_1 = 0$$
$$u(x_2) = u_2 + H(x_2)a_1 = \frac{\bar{u}}{2} + (+1)\left(-\frac{\bar{u}}{2}\right) = 0$$
$$u(x_3) = u_3 + H(x_3)a_3 = \frac{\bar{u}}{2} + (-1)\left(-\frac{\bar{u}}{2}\right) = \bar{u}$$
$$u(x_4) = \bar{u} \quad (55)$$

This result matches the exact solution:

$$u_1(x_1) = 0, u_2(x_2) = 0, u_3(x_3) = \bar{u}, u_4(x_4) = \bar{u}$$

Compare the result between case 1 and case 3 as follow

Case 3



$$u_1 = 0$$
$$u_2 = \bar{u}/2$$
$$u_3 = \bar{u}/2$$
$$u_4 = \bar{u}$$
$$a_1 = -\bar{u}/2$$

$$a_2 = -\bar{u}/2 \Rightarrow \begin{cases} u(x_1) = u_1 = 0 \\ u(x_2) = \frac{\bar{u}}{2} + (+1)\left(-\frac{\bar{u}}{2}\right) = 0 \\ u(x_3) = \frac{\bar{u}}{2} + (-1)\left(-\frac{\bar{u}}{2}\right) = \bar{u} \\ u(x_4) = \bar{u} \end{cases}$$

Comments: In the comparison between Case 1 and Case 3, although the enriched degrees of freedom a_1 and a_2 differ in sign, the actual displacements at the nodes remain unchanged. This confirms that the numerical solution is invariant with respect to the sign of the enrichment terms. Across all three cases, the numerical results-particularly the nodal displacements-are consistent. This indicates that the solution of the discontinuity problem is independent of the specific form or sign of the Heaviside function used in the enrichment process.

4. Conclusions

This study examined the influence of different Heaviside function forms on numerical results in discontinuity problems, focusing on structural cases involving cracks and material separations. Through comparative analysis of various formulations, it was demonstrated that the choice of Heaviside function does not affect key results such as displacement fields or overall structural response. Although differences in enriched degrees of freedom–particularly in sign–were observed, these did not alter the final physical results.

The results confirm that the numerical implementation is robust and that standard Heaviside functions can be used interchangeably without compromising accuracy or reliability. This provides practical flexibility in selecting enrichment functions and supports the simplification of computational procedures in discontinuity modeling. Ultimately, the results contribute to improving the efficiency of numerical strategies while maintaining confidence in the consistency of structural analysis results.

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