Journal of Materials and Construction Vol 15 No.01 (2025)

Journal of Materials and Construction
ISSN 2734-9438
Website: www. jomc.vn

FEM-based prediction of elastic critical loads in steel frames with nonlinear semi-rigid connections

Quoc Anh Vu[1], Hai Quang Nguyen[2], Dung Bao Trung Le[1*]

[1]Faculty of Civil Engineering, Hanoi Architectural University

[2]Faculty of Mechanical-Automotive and Civil Engineering, Electric Power University

KEYWORDS

Steel Frames
Semi-Rigid Connection
Elastic Stability
Nonlinear Analysis
Critical Load

ABSTRACT

Due to the manufacturing constraints, steel frames are often divided into components and assembled together through connections. Studies have shown that the joints between components in steel frames are often semi-rigid. In addition, steel frames typically consist of slender members, so the frame tends to undergo large displacements, making the influence of the P-Delta effect on the steel frame more pronounced. These issues significantly affect the global stability behavior of the frame and must be taken into account. This paper presents a new elastic stability analysis method for steel frames with semi-rigid connections based on the finite element method. The method utilizes a novel stiffness matrix formulation and an equivalent nodal load vector, combined with a load-increment strategy to solve the nonlinear equilibrium equations. The relationship between displacement and load is investigated, the critical load is determined. An algorithm flowchart is provided, and verification examples are conducted. Comments on the instability process of frames with different types of connections are also presented. This paper is applicable to the study and analysis of the elastic stability of steel frames with nonlinear semi-rigid connections in particular, and rigid or semi-rigid connections in general, taking into account the influence of the P-Delta effect.

1. Introduction

Structural stability refers to the ability of a structure to maintain its original position or initial equilibrium under deformation corresponding to applied loads. The problem of structural stability has been studied since early times. Beginning with Euler's solution for determining the critical load of columns with hinged ends, first published in 1744 [1], [2], numerous studies and technical documents have addressed stability analysis for frame structures. Design standards in both developed countries and Vietnam also require an overall stability assessment for frame systems. Section C2 of the American steel design standard AISC-LRFD (1999) [3], Section 5.2.2 of the European standard EN 1993-1-1 [4], and Section 4.2.6 of the Vietnamese standard TCVN 5575:2012 [5] all emphasize the necessity of stability analysis for steel frames. According to [1], there are five main approaches to stability analysis for frame systems. This paper presents a Finite Element Method (FEM)-based approach for analyzing elastic static stability through critical load calculation.

Research on the stability of frame structures with rigid connection elements has been conducted by authors such as Trinh and Binh (2005) [2], and Bazant and Cedolin (2010) [6]; while studies on frame systems with linear semi-rigid connections include works by Anh, V.Q. (2001) [7], Anh, V.Q. (2002) [8], and Galambos and Surovek (2008) [9]. A characteristic of these frame structures is that both the linear elastic stiffness matrix and the geometric stiffness matrix contain constant coefficients. The general approach to determining the critical load is to establish and solve the stability equation. This equation can be solved either by expanding the determinant and solving the resulting higher-order equation or by using iterative methods. The advantage of this approach lies in its low computational complexity. However, it has several limitations: the load must be applied at the nodes, moment effects are neglected, the P–Delta effect is not considered, and the deformation diagram of the frame cannot be updated in the case of large displacement problems.

As stated in [10], there are seven methods for establishing the linear stiffness matrix and geometric stiffness matrix of semi-rigid beam–column elements. These methods serve as important foundations for developing stability analysis approaches for steel frames with semi-rigid connections using the Finite Element Method (FEM). In the current literature, no stability studies have been found that apply the first, fourth, and fifth methods.

The second method, which involves assembling springs to form a hybrid element, has been used in stability analyses by Lui and Chen (1986) [11], Lui and Chen (1988) [12], and Chan and Chui (2000) [13].

The third method, which modifies the rotation angle of beam ends in the slope–deflection expression of a beam element with rigid ends to account for semi-rigid connections, has been applied in studies by Chen and Lui (1987) [14], Kim and Choi (2001) [15], Pinheiro and Silveira (2005) [16], and Chen and Lui (2018) [17].

The sixth method involves developing the displacement function

*Corresponding author: Trungldb@hau.edu.vn
Received 06/05/2025, explanation 20/05/2025, accepted 21/05/2025
Link DOI: https://doi.org/10.54772/jomc.v15i01.960

for semi-rigid beam–column elements by adjusting the end rotations in the displacement formulation based on third-order Hermitian functions and the nodal displacement vector of a rigid-end beam element. This method was employed in the work of Sekulovic and Salatic (2001) [18].

The seventh method, which constructs shape functions using the unit displacement method for stability analysis, has been used by Anh, V.Q. [7], [8], and Igić, Zdravković, Zlatkov, Živković, and Stojić [19].

All studies [11-19] were conducted on steel frames with semi-rigid connections using nonlinear modeling approaches. For steel frames with this type of connection, the moments and axial forces, the instantaneous stiffness of the connections, and the loads on the span are nonlinearly interrelated. As a result, the linear elastic stiffness matrix and the geometric stiffness matrix also depend on these nonlinear interactions. During the analysis process, the coefficients of these stiffness matrices are not constant but vary continuously, making it impossible to solve the stability equation directly. Instead, it is necessary to solve a nonlinear system of equilibrium equations. This analytical approach is also known as stability analysis considering the P–Delta effect [8].

This paper presents an elastic stability analysis method based on the Finite Element Method (FEM), which utilizes linear elastic stiffness matrices, geometric stiffness matrices, and converted nodal load vectors developed by Anh, Quang, and Trung (2025) in [10]. An incremental load strategy is employed to calculate the ultimate load for steel frames with nonlinear semi-rigid connections. The algorithm flowchart and verification examples are provided.

While the method of solving stability equations can only determine the critical load at the onset of instability, the approach of analyzing stability by solving equilibrium equations enables simulation of the entire load–lateral displacement response of the frame until it becomes unstable. Therefore, this method can be used to study the instability progression of planar steel frames, as well as to compare the instability behavior of planar steel frames with different types of connections. In addition to the nonlinear semi-rigid connection model, the method can also be applied to rigid connection models and linear or multi-linear semi-rigid connection models.

2. The semi-rigid beam-column finite element model

2.1. The semi-rigid beam-column stiffness matrices

In this article, the semi-rigid connections are modeled by rotational springs and are integrated into the beam-column element as a single finite element. Consider a beam-column element e^{th}, where the connections at ends A and B are elastic rotational springs, forming a semi-rigid beam-column element. The local coordinate system Oxy has the x-axis aligned with the longitudinal axis of the element, the y-axis perpendicular to the element's longitudinal axis and pointing upward, with the origin of coordinates at end A. The symbols A, I, L, and E represent the cross-sectional area, moment of inertia, length, and elastic modulus of the material, respectively. In the i^{th} loading step, the symbols N_A, Q_A and M_A represent the longitudinal force (in the x-axis direction), shear force (in the y-axis direction), and bending moment (rotation around the z-axis) at end A, respectively, and similarly for N_B, Q_B and M_B at end B.

The displacements u_A, v_A, and u_B, v_B represent the axial and transverse displacements of nodes A and B, respectively. The rotational displacements around the z-axis of the frame nodes at A and B are denoted by θ_A and θ_B, while θ_{eA} and θ_{eB} are the rotation angles at the beam ends. The connection rotational angles at nodes A and B are denoted as θ_{cA} and θ_{cB}, respectively. The rotational stiffnesses of the connections at nodes A and B are denoted by k_A and k_B, respectively.

Fig. 1 illustrates the kinematic relationship among displacement, internal force, and deformation of an element with a semi-rigid connection. Assuming that the semi-rigid connection is dimensionless and that the effects of axial and shear forces on the connection's behavior are negligible, the element material is considered linearly elastic, the beam behavior follows the Euler-Bernoulli model, and the connection's moment-rotation relationship can be linear, piecewise-linear, or nonlinear.

The nodal displacement vector of the e^{th} element, at the i^{th} loading step, in the local coordinate system is:

$$\{\delta_s\}_e = \{u_A \quad v_A \quad \theta_{eA} \quad u_B \quad v_B \quad \theta_{eB}\}^T \quad (1)$$

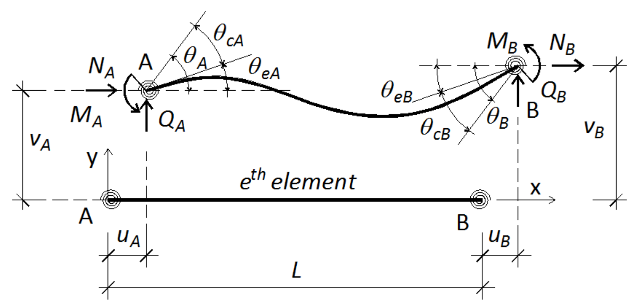


Figure 1. Semi-rigid beam-column element.

Here, the relationship between the rotation angles at the beam with semi-rigid connections at ends has the form:

$$\begin{cases} \theta_{eA} = \theta_A - \theta_{cA} \\ \theta_{eB} = \theta_B - \theta_{cB} \end{cases} \quad (2)$$

In each loading step, the stiffness of the connection, element length, and the geometric and material properties of the element are assumed to remain unchanged. The shape functions corresponding to the displacements of a semi-rigid beam-column element have been developed in [10].

The stiffness matrix of a semi-rigid beam-column element, which includes both the linear elastic stiffness matrix and the geometric stiffness matrix, is derived from the deformation potential energy expression of the semi-rigid beam-column element $U_{s,e}$.

Let $\{P_s\}_e$ be the nodal load vector in the local coordinate system, $[k_{L-Es}]_e$ the linear elastic stiffness matrix, and $[k_{Gs}]_e$ the geometric stiffness matrix of the semi-rigid beam-column element. By applying Castigliano's theorem (Part 1), the relationship between load, deformation potential energy, and displacement of the e^{th} element is obtained as follows:

$$\{P_s\}_e = \frac{\partial U_{s,e}}{\partial \delta_s} = [k_s]_e\{\delta_s\}_e \quad (3)$$

here,

$$[k_s]_e = [k_{L-Es}]_e + [k_{Gs}]_e \quad (4)$$

represents stiffness matrix of the element with two semi-rigid connection ends.

The linear elastic stiffness matrix of the semi-rigid beam-column element established in [10] is as follows:

$$[k_{L-Es}]_e = \begin{bmatrix} k_{L-Es}^{1,1} & 0 & 0 & k_{L-Es}^{1,4} & 0 & 0 \\ 0 & k_{L-Es}^{2,2} & k_{L-Es}^{2,3} & 0 & k_{L-Es}^{2,5} & k_{L-Es}^{2,6} \\ 0 & k_{L-Es}^{3,2} & k_{L-Es}^{3,3} & 0 & k_{L-Es}^{3,5} & k_{L-Es}^{3,6} \\ k_{L-Es}^{4,1} & 0 & 0 & k_{L-Es}^{4,4} & 0 & 0 \\ 0 & k_{L-Es}^{5,2} & k_{L-Es}^{5,3} & 0 & k_{L-Es}^{5,5} & k_{L-Es}^{5,6} \\ 0 & k_{L-Es}^{6,2} & k_{L-Es}^{6,3} & 0 & k_{L-Es}^{6,5} & k_{L-Es}^{6,6} \end{bmatrix} \quad (5)$$

wherein:

$$k_{L-Es}^{1,1} = k_{L-Es}^{4,4} = -k_{L-Es}^{1,4} = -k_{L-Es}^{4,1} = \frac{EA}{L}$$

$$k_{L-Es}^{2,2} = k_{L-Es}^{5,5} = -k_{L-Es}^{2,5} = -k_{L-Es}^{5,2} = 12EI(3d_2^2L - 3d_2d_3L^2 + d_3^2L^3) + 36E^2I^2\left(\frac{d_2^2}{k_A} + \frac{d_8^2}{k_B}\right)$$

$$k_{L-Es}^{2,3} = k_{L-Es}^{3,2} = -k_{L-Es}^{3,5} = -k_{L-Es}^{5,3} = 6EI(-3d_2^2L^2 + 2d_2d_3L^3 + 2d_2d_5L - d_3d_5L^2) + 12E^2I^2\left(\frac{d_2d_5}{k_A} + \frac{d_7d_8}{k_B}\right)$$

$$k_{L-Es}^{2,6} = k_{L-Es}^{6,2} = -k_{L-Es}^{5,6} = -k_{L-Es}^{6,5} = 6EI(2d_2d_7L - 3d_2d_8L^2 - d_3d_7L^2 + 2d_3d_8L^3) + 12E^2I^2\left(\frac{d_2d_7}{k_A} + \frac{d_8d_9}{k_B}\right)$$

$$k_{L-Es}^{3,3} = 4EI(3d_2^2L^3 - 3d_2d_5L^2 + d_5^2L) + 4E^2I^2\left(\frac{d_5^2}{k_A} + \frac{d_7^2}{k_B}\right)$$

$$k_{L-Es}^{3,6} = k_{L-Es}^{6,3} = 2EI(-3d_2d_7L^2 + 6d_2d_8L^3 + 2d_5d_7L - 3d_5d_8L^2) + 4E^2I^2\left(\frac{d_5d_7}{k_A} + \frac{d_7d_9}{k_B}\right)$$

$$k_{L-Es}^{6,6} = 4EI(d_7^2L - 3d_7d_8L^2 + 3d_8^2L^3) + 4E^2I^2\left(\frac{d_7^2}{k_A} + \frac{d_9^2}{k_B}\right).$$

And the geometric stiffness matrix of the semi-rigid beam-column element established in [10]:

$$[k_{Gs}]_e = N\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{Gs}^{2,2} & k_{Gs}^{2,3} & 0 & k_{Gs}^{2,5} & k_{Gs}^{2,6} \\ 0 & k_{Gs}^{3,2} & k_{Gs}^{3,3} & 0 & k_{Gs}^{3,5} & k_{Gs}^{3,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{Gs}^{5,2} & k_{Gs}^{5,3} & 0 & k_{Gs}^{5,5} & k_{Gs}^{5,6} \\ 0 & k_{Gs}^{6,2} & k_{Gs}^{6,3} & 0 & k_{Gs}^{6,5} & k_{Gs}^{6,6} \end{bmatrix} \quad (6)$$

wherein:

$$k_{Gs}^{2,2} = k_{Gs}^{5,5} = -k_{Gs}^{2,5} = -k_{Gs}^{5,2} = d_1^2L + 6d_1d_2L^2 - 2d_1d_3L^3 + 12d_2^2L^3 - 9d_2d_3L^4 + \frac{9}{5}d_3^2L^5$$

$$k_{Gs}^{2,3} = k_{Gs}^{3,2} = -k_{Gs}^{3,5} = -k_{Gs}^{5,3} = -d_1d_2L^3 - d_1d_4L + d_1d_5L^2 - \frac{9}{2}d_2^2L^4 + \frac{9}{5}d_2d_3L^5 - 3d_2d_4L^2 + 4d_2d_5L^3 + d_3d_4L^3 - \frac{3}{2}d_3d_5L^4$$

$$k_{Gs}^{2,6} = k_{Gs}^{6,2} = -k_{Gs}^{5,6} = -k_{Gs}^{6,5} = d_1d_6L + d_1d_7L^2 - d_1d_8L^3 + 3d_2d_6L^2 + 4d_2d_7L^3 - \frac{9}{2}d_2d_8L^4 - d_3d_6L^3 - \frac{3}{2}d_3d_7L^4 + \frac{9}{5}d_3d_8L^5$$

$$k_{Gs}^{3,3} = \frac{9}{5}d_2^2L^5 + 2d_2d_4L^3 - 3d_2d_5L^4 + d_4^2L - 2d_4d_5L^2 + \frac{4}{3}d_5^2L^3$$

$$k_{Gs}^{3,6} = k_{Gs}^{6,3} = -d_2d_6L^3 - \frac{3}{2}d_2d_7L^4 + \frac{9}{5}d_2d_8L^5 + d_4d_6L - d_4d_7L^2 + d_4d_8L^3 + d_5d_6L^2 + \frac{4}{3}d_5d_7L^3 - \frac{3}{2}d_5d_8L^4$$

$$k_{Gs}^{6,6} = d_6^2L + 2d_6d_7L^2 - 2d_6d_8L^3 + \frac{4}{3}d_7^2L^3 - 3d_7d_8L^4 + \frac{9}{5}d_8^2L^5.$$

Here, the parameters in the Eq. (5, 6) are as follows:

$$d_0 = 12E^2I^2 + 4EILk_A + 4EILk_B + L^2k_Ak_B,\ d_1 = \frac{6EI(2EI+Lk_B)}{Ld_0},\ d_2 = \frac{(2EI+Lk_B)k_A}{Ld_0},\ d_3 = \frac{2(EIk_A+EIk_B+Lk_Ak_B)}{L^2d_0},\ d_4 = \frac{L(4EI+Lk_B)k_A}{d_0},\ d_5 = \frac{2(3EI+Lk_B)k_A}{d_0},$$
$$d_6 = \frac{2LEIk_B}{d_0},\ d_7 = \frac{Lk_Ak_B}{d_0},\ d_8 = \frac{(2EI+Lk_A)k_B}{Ld_0},\ d_9 = \frac{2(3EI+Lk_A)k_B}{d_0} \quad (7)$$

2.2. The equivalent nodal load vector

Consider a semi-rigid beam element subjected to a uniformly distributed load and concentrated loads, as illustrated in Fig. 2. The equivalent nodal load vector in the local coordinate system, derived according to the finite element method (FEM), has been established in [10].

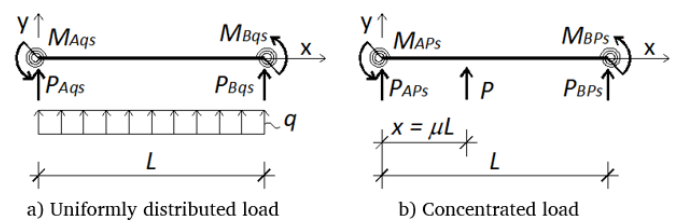


Figure 2. The load on the semi-rigid beam element is converted to nodes.

In case of the uniformly distributed load $q(x) = q$ as shown in Fig. 2a, the equivalent nodal load vector $\{P_{qs}\}_e$ is as follows:

$$\{P_{qs}\}_e = \{P_{Aqs} \quad M_{Aqs} \quad P_{Bqs} \quad M_{Bqs}\}_e^T = q\begin{bmatrix} -\frac{d_1L^2}{2} - d_2L^3 + \frac{d_3L^4}{4} + L \\ \frac{d_2L^4}{4} + \frac{d_4L^2}{2} - \frac{d_5L^3}{3} \\ \frac{d_1L^2}{2} + d_2L^3 - \frac{d_3L^4}{4} \\ -\frac{d_6L^2}{2} - \frac{d_7L^3}{3} + \frac{d_8L^4}{4} \end{bmatrix} \quad (8)$$

And in case of concentrated load P at location $x = \mu L$ as shown in Fig. 2b, the equivalent nodal load vector $\{P_{Ps}\}_e$ has the form:

$$\{P_{Ps}\}_e = \{P_{APs} \quad M_{APs} \quad P_{BPs} \quad M_{BPs}\}_e^T = P\begin{bmatrix} 1 - d_1\mu L - 3d_2\mu^2L^2 + d_3\mu^3L^3 \\ d_4\mu L - d_5\mu^2L^2 + d_2\mu^3L^3 \\ d_1\mu L + 3d_2\mu^2L^2 - d_3\mu^3L^3 \\ -d_6\mu L - d_7\mu^2L^2 + d_8\mu^3L^3 \end{bmatrix} \quad (9)$$

The parameters in Eq. (8, 9) are determined in Eq. (7).

3. The method of elastic stability analysis of steel frames with nonlinear semi-rigid connections

From Eq. (3), the equilibrium equation of element e^{th} in local coordinate system has the form:

$$\{P_s\}_e = [k_s]_e\{\delta_s\}_e \quad (10)$$

From Eq. (12), and following the transformation principles of the finite element method as described in [10, 20], the equilibrium equation of the steel frame in the global coordinate system is obtained:

$$[K_s^*]\{\delta_s^*\} = \{P_s^*\} \quad (11)$$

In Eq. (11), $[K_s^*]$, $\{\delta_s^*\}$, and $\{P_s^*\}$ denote the global stiffness matrix, nodal displacement vector, and equivalent nodal load vector, respectively, after incorporating the boundary conditions.

Here,

$$[K_s^*] = [K_{L-Es}^*] + [K_{Gs}^*] \quad (12)$$

where $[K_{L-Es}^*]$ and $[K_{Gs}^*]$ represent the linear elastic stiffness matrix and geometric stiffness matrix of the steel frame with semi-rigid connections in the global coordinate system, respectively, after applying the boundary conditions. These matrices are obtained from the stiffness matrices presented in Eqs. (5) and (6), using the finite element method.

When the steel frame is composed of elements with ends that are either rigid, hinged, or semi-rigid with constant rotational stiffness, the coefficients of the matrices $[K_{L-Es}^*]$ and $[\overline{K}_{Gs}^*]$ remain constant. To determine the critical force approximately, it is assumed that the loads acting on the system are applied exclusively at the nodes. Since the axial forces P are conventionally considered inherent characteristics of the system rather than external loads, the equivalent nodal load vector is taken to be zero $\{P_s^*\} = \{0\}$. By substituting Eq. (12) into Eq. (11), the global equilibrium equation becomes:

$$[K_s^*]\{\delta_s^*\} = ([K_{L-Es}^*] + [K_{Gs}^*])\{\delta_s^*\} = \{0\} \quad (13)$$

According to the stability criterion for static equilibrium, the system becomes unstable when there is a state that deviates from the initial state; that is, when

$$\{\delta_s^*\} \neq \{0\}.$$

The system will be unstable when the determinant of the global stiffness matrix is zero:

$$det|K_s^*| = det|K_{L-Es}^* + K_{Gs}^*| = 0 \quad (14)$$

In Eq. (14), the matrix $[K_{Gs}^*]$ is defined with respect to the forces N. If we call

$$[\widehat{K}_{Gs}^*]$$

the corresponding matrix established according to the reference force N_c, chosen arbitrarily, or $N_c = 1$, and set $N = \lambda N_c$, then in the case of small displacements, Eq. (14) takes the form:

$$det|K_s^*| = det|K_{L-Es}^* + \lambda \widehat{K}_{Gs}^*| = 0 \quad (15)$$

Eq. (15) is the *stability equation*, where λ represents the eigenvalues to be found. Solve the stability equation to find the minimum value, called the critical parameter, λ_{cr}. There are many ways to do this: by solving higher-order equations to find eigenvalues, thereby determining λ_{cr}; or by iteratively solving to directly determine λ_{cr}. From the critical parameter, the corresponding critical load, P_{cr}, can be determined.

When a steel frame consists of nonlinear semi-rigid connections, the coefficients of matrices $[K_{L-Es}^*]$ and $[\overline{K}_{Gs}^*]$ are no longer constant, but are continuously changing. They depend on the instantaneous stiffness of the connection, axial force, and even the deformation scheme of the structural system (or P-Delta effect [8]). This problem cannot be solved by simply solving the stability equation, but must be addressed directly by solving the equilibrium equation of the structural system according to Eq. (11). This equation involves two issues: connection nonlinearity and geometric nonlinearity. A simple loading increment strategy, similar to the incremental single-step method [14, 21], can be applied to solve this system of nonlinear equilibrium equations.

The symbols N_0, N_1, and N_{max} represent the axial force from the previous calculation, the axial force in the next calculation, and the axial force in the element with the largest change in axial force after each calculation step, respectively.

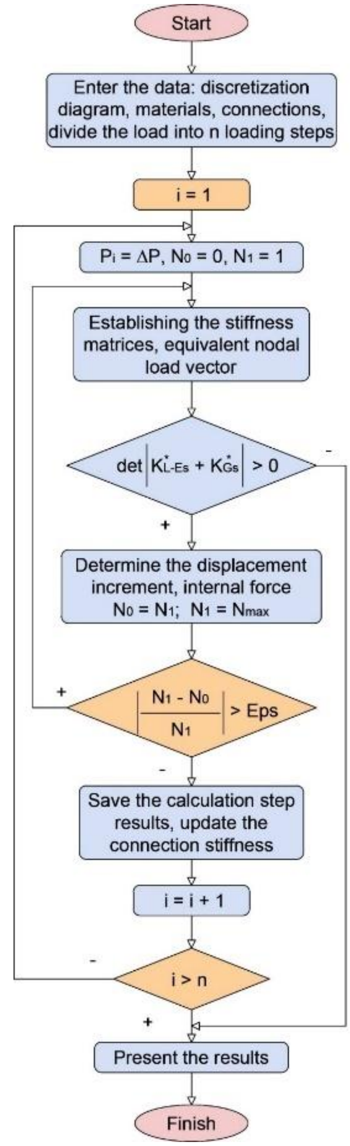


Figure 3. The elastic stability analysis of semi-rigid steel frame flowchart.

At the first load increment step, the stiffness of the connections is assumed to be equal to the initial stiffness, and the initial axial force N_0 is taken as zero. The incremental load steps ΔP are taken equally and small enough to ensure accuracy. After each load increment i^{th}, the displacements, stiffnesses of the connections, and axial forces are calculated and updated. The increased load causes the axial force and rotation angle in the connection to increase, and the stiffness matrix of the structural system to decrease. The symbol *Eps* represents the threshold variation in axial force between two successive calculation steps. The system becomes unstable when the determinant value in Eq. (14) becomes negative. The load increase process stops, and the load at this point is the critical load P_{cr}. A brief flowchart of the methodology is presented in Fig. 3.

4. Numerical Verification and Discussion

Numerical examples are conducted to verify the reliability of the stiffness matrices and the proposed method for stability analysis of steel frames with semi-rigid connections. These examples are compared with SAP2000 software results and published research to validate both the accuracy and applicability of the approach. Validation is performed for the load-lateral displacement relationship during load increments and for the elastic critical load, determined at the peak point of the load-lateral displacement curve.

This example evaluates and verifies the load-lateral displacement (P-U) relationship and the elastic critical load, accounting for the P-Delta effect, by comparing the proposed method with SAP2000 results. The verification is carried out on two steel frames. The first is a two-storey, single-span steel frame with wide- flange columns of section W12 × 96 and beams of section W14 × 48, as shown in Fig. 4. The second is a three-storey, single-span steel frame with wide-flange columns of section W10 × 88 and beams of section W12 × 45, as shown in Fig. 5.

Beam-column connections in both frames are analyzed under two conditions: rigid connections and linear semi-rigid connections with a stiffness of k = 12428.3 kN.m/rad. Column connections, where applicable, are modeled as rigid, while column bases are assumed to be hinged. All column and beam elements are made of steel with a modulus of elasticity E = 199948 MPa. The material is assumed to behave elastically throughout the analysis.

The calculation results based on the proposed theory are obtained using a self-developed Matlab program, while the frame stability analysis is carried out using SAP2000 software, incorporating the P-Delta effect. The external load on the frame is uniformly increased until the structure reaches an unstable state. The maximum variation in axial force is observed in the first-floor column on the right side. A limit variation in axial force between two successive calculation steps is set as Eps = 0.000001.

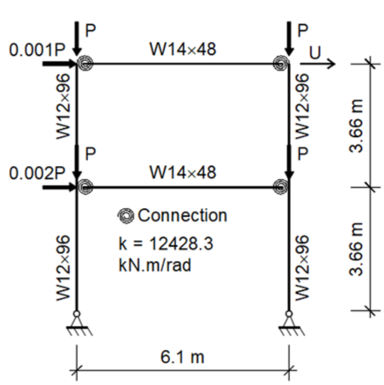


Figure 4. The two-story steel frame.

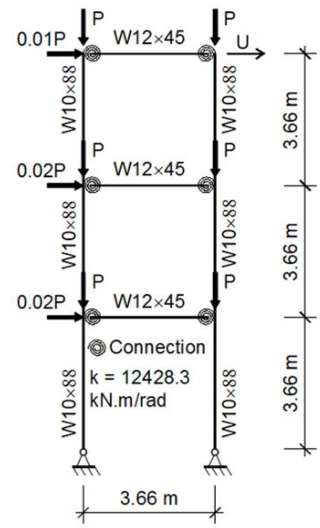
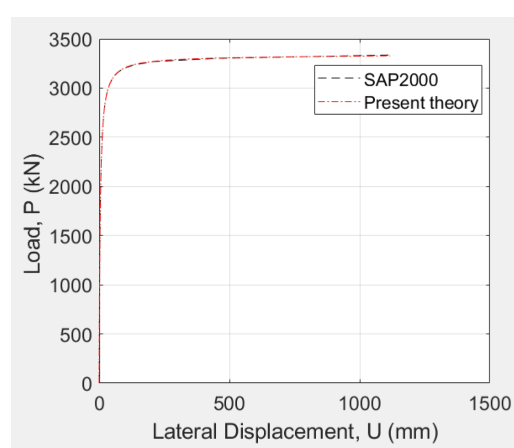


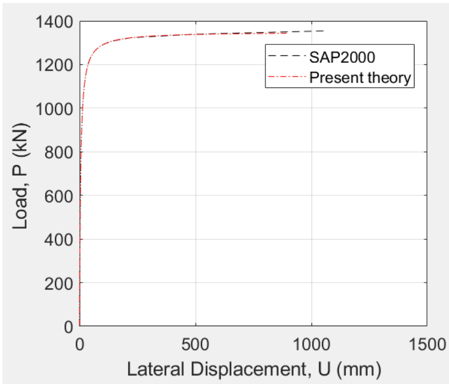
Figure 5. The three-story steel frame.

The analysis results for the two-storey frame are illustrated in Fig. 6, while those for the three-storey frame are shown in Fig. 7. In each figure, the hidden black line represents the SAP2000 results, and the central red line represents the results from the present theory. A comparison of the curves indicates that the load–lateral displacement relationships for all four cases closely coincide, validating the accuracy of the proposed theoretical approach.

The shape of the load–lateral displacement curves demonstrates that the stiffness of the steel frames deteriorates rapidly as the structure approaches its critical state. This degradation is more pronounced in frames with rigid connections, as evident from sharp breaks and slope changes in the curves, particularly in Fig. 6(a) and Fig. 7(a). In contrast, frames with semi-rigid connections exhibit a more gradual and continuous reduction in stiffness, as shown in Fig. 6(b) and Fig. 7(b). These findings highlight the distinct elastic stability behavior of rigid and semi-rigid steel frames when the P-Delta effect is considered.

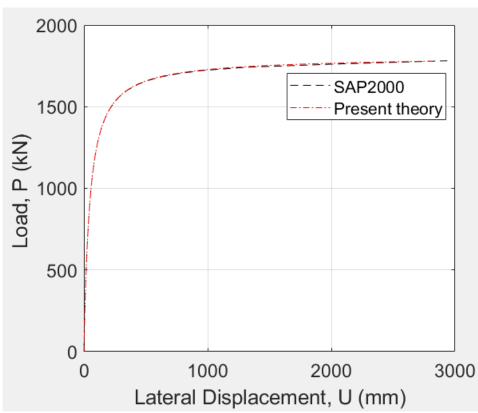


(a) Rigid beam-column connection

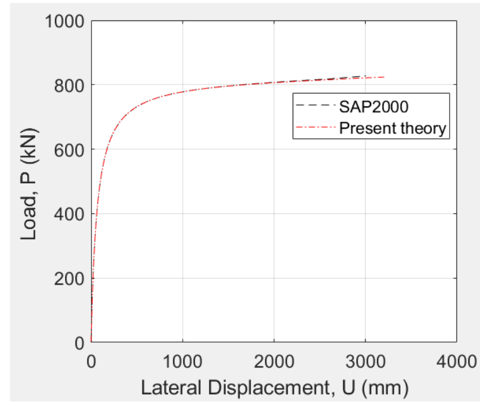


(b) Semi-rigid beam-column connection

Figure 6. Comparison of P-U load-lateral displacement curves for two-story frame for verification study.



(a) Rigid beam-column connection



(b) Semi-rigid beam-column connection

Figure 7. Comparison of P-U load-lateral displacement curves for three-story frame for verification study.

The elastic critical loads of the steel frames P_{cr} are determined at the peak points of the load–lateral displacement curves and are summarized in Table 1. The data presented in the table indicate that the critical loads obtained from all analyzed cases are nearly identical. The slight discrepancy between the results of the present theoretical method and those from SAP2000 software is attributed to the acceptable tolerance defined for terminating the iterative calculation.

As expected, the elastic critical load of a steel frame with rigid connections is consistently higher than that of a frame with semi-rigid connections. The close agreement between the load–displacement curves and the computed elastic critical loads confirms the accuracy and reliability of the proposed theoretical approach.

Table 1. The elastic critical load of steel frames, P_{cr} (kN).

Analysis case	The two-story frame		The three-story frame	
	Rigid connection	Semi-rigid connection	Rigid connection	Semi-rigid connection
SAP2000 (A)	3350.1	1358.9	1793.3	824.6
Present theory (B)	3350.1	1345.4	1793.3	821.3
(B-A/A)×100%	0 %	-0.010 %	0 %	-0.004 %

5. Conclusions

This paper presents a modern method for the analysis of the elastic stability of steel frames with nonlinear semi-rigid connections. This method is established based on novel stiffness matrices and an equivalent nodal load vector, combined with a single-step load-increment strategy to determine the critical load.

This analysis method does not involve solving classical stability equations but instead requires solving equilibrium equations. Its reliability and practical applicability are validated through numerical examples. Since it is derived from the solution of the equilibrium equations, this stability analysis method accounts for nonlinear connection stiffness, P-Delta effects, span-applied loads, and can update the geometry when necessary. Therefore, stability calculations using this method provide more realistic and accurate results than traditional stability analysis techniques. In the context of analyzing the stability of steel frames with nonlinear semi-rigid connections considering the P-Delta effect, due to the nonlinear interdependence between moment, axial force, connection stiffness, and geometric effects, solving a nonlinear equilibrium equation system to determine the critical load is essential.

The present study demonstrates that the stability behavior of steel frames varies depending on the type of connection. Steel frames with nonlinear semi-rigid connections exhibit the greatest flexibility up to the critical load, while frames with rigid connections are stiffer and undergo a more rapid degradation in stiffness as the load approaches the critical threshold.

Although the proposed stability analysis method that includes the P-Delta effect is computationally demanding, it is necessary for accurately analyzing steel frames with semi-rigid connections. By integrating semi-rigid connections into the element formulation to form a hybrid element, as presented in this study, the number of elements required in the computation is reduced, thereby enhancing computational efficiency.

References

[1]. Galambos, T.V., *Guide to stability design criteria for metal structures,* 4th edition. New York, USA: John Wiley & Son, 1976.

[2]. Trinh, L.T., Binh, Đ.V., *Structural Stability*. Hanoi, Vietnam: Science and Technics Publishing House, 2005.

[3]. *AISC-LRFD Load and Resistance Factor Design Specification for Structural Steel Buildings*. Chicago, Illinois, USA: American Institute of Steel Construction, Inc, 1999.

[4]. *EN 1993-1-1 Design of steel structures – Part 1-1: General rules and rules for buildings*. Brussels, Belgium: CEN, 2005.

[5]. *TCVN 5575:2024 Kết cấu thép - Tiêu chuẩn thiết kế*. Hanoi, Vietnam: Ministry of Science and Technology, 2024.

[6]. Bazant, Z. P., Cedolin, L., *Stability of structures: elastic, inelastic, fracture and damage theories*. Singapore: World Scientific. Doi:10.1142/9789814317047, 2010.

[7]. Anh, V.Q., Stability anaysis of steel frames with semirigid connections and semirigid connections with rigidzones. *Vietnam Journal of Mechanics*, Vol. 23, no. 3, 134-148, 2001.

[8]. Anh, V.Q., Stability anaysis of steel frames with semi-rigid connections and rigidzones by using P-Delta effect. *Vietnam Journal of Mechanics*, Vol. 24, no. 1, 14-24, 2002.

[9]. Galambos, T.V., Surovek, A.E., *Structural stability of steel: concepts and applications for structural engineering*. New Jersey, USA: John Wiley & Son, Inc, 2008.

[10]. Anh, V.Q., Trung, L.D.B., Quang, N.H., Shape Functions Development for Beam-Column Element with Semi-Rigid Connections in Second-Order Steel Frame Analysis. Tehran, Iran: *Civil Engineering Journal (C.E.J)*, Vol.11, No. 01: 369-392, 2025. Doi: http://dx.doi.org/10.28991/CEJ-2025-011-01-021.

[11]. Lui, E. M., Chen, W. F., Analysis and behaviour of flexibly-jointed frames. *Engineering Structures*, 8(2), 107–118, 1986. Doi: 10.1016/0141-0296(86)90026-X.

[12]. Lui, E. M., Chen, W. F., Behavior of braced and unbraced semi-rigid frames. *International Journal of Solids and Structures*, 24(9), 893–913, 1988. Doi: 10.1016/0020-7683(88)90040-6.

[13]. Chan, S. L., Chui, P. T., *Non-linear static and cyclic analysis of steel frames with semi-rigid connections*. Amsterdam, Netherlands: Elsevier, 2000.

[14]. Chen, W. F., Lui, E. M., *Structural stability theory and implementation*. New York, USA: Elsevier, 1987.

[15]. Kim, S. E., Choi, S. H., Practical advanced analysis for semi-rigid space frames. *International Journal of Solids and Structures*, 38(50–51), 9111–9131, 2001. Doi:10.1016/S0020-7683(01)00141-X.

[16]. Pinheiro, L., Silveira R.A. M., Computational procedures for nonlinear analysis of frames with semi-rigid connections. *Latin American Journal of Solids and Structures*, 2, 339-367, 2005.

[17]. Chen, W. F., Lui, E. M., *Stability Design of Steel Frames,* Reissued. New York, USA: CRC Press, Inc, 2018.

[18]. Sekulovic, M., Salatic, R., Nonlinear analysis of frames with flexible connections. *Computers and Structures*, 79(11), 1097–1107, 2001. Doi:10.1016/S0045-7949(01)00004-9.

[19]. Igić, T., Zdravković, S., Zlatkov, D., Živković, S., Stojić, R., Stability design of structures with semi-rigid connections. Facta Universitatis: *Architecture and Civil Engineering*, 8(2), 261-275, 2010.

[20]. Przemieniecki, J. S., *Theory of matrix structural analysis*. North Chelmsford, United States: Courier Corporation, 1985.

[21]. McGuire, W., Gallagher, R. H., Saunders, H., *Matrix Structural Analysis*. Hoboken, United States: John Wiley & Sons, 2014.